CSI 4105 Fall 2002 **Assignment #1: Solutions**

1. Asymptotic notation properties

a. TRUE

$g_1 \in O(f_1) \rightarrow \exists n_1, c_1$ such that $g_1(n) \leq c_1 f(n)$ for all $n \geq n_1$.

$g_2 \in O(f_2) \rightarrow \exists n_2, c_2$ such that $g_2(n) \leq c_2 f(n)$ for all $n \geq n_2$.

Thus, $(g_1 \cdot g_2)(n) = g_1(n) + g_2(n) \leq (c_1 + c_2) f(n)$ for all $n \geq max(n_1, n_2)$.

So, for $c = c_1 + c_2$, $n_0 = max(n_1, n_2)$,

$(g_1 \cdot g_2)(n) \leq cf(n)$, for all $n \geq n_0$.

b. TRUE

$g \in O(f)$

$\Leftrightarrow \exists c, n_0$ such that $g(n) \leq cf(n)$ for all $n \geq n_o$

$\Leftrightarrow \exists c, n_0$ such that $f(n) \geq \frac{1}{c} g(n)$ for all $n \geq n_0$

$\Leftrightarrow \exists c', n_0$ (with $c' = \frac{1}{c}$) such that $f(n) \geq c' g(n)$ for all $n \geq n_0$

$\Leftrightarrow f \in \Omega(g)$.

c. FALSE

For $f(n) = 2^n$ ($f(\frac{n}{2}) = 2^{\frac{n}{2}} = (\sqrt{2})^n$), we have $2^n \notin \Theta(\sqrt{2}^n)$.

$2^n \notin \Theta(\sqrt{2}^n)$ since $2^n \notin O(\sqrt{2}^n)$.

If it were, then $\exists n_0, c$ such that $\frac{2^n}{\sqrt{2}^n} = \sqrt{2}^n \leq c$ for $n \geq n_0$, which is impossible.

d. TRUE

If $g \in o(f)$, then $g \in O(f)$.

So, $\exists c, n_0$ such that $g(n) \leq cf(n)$ for all $n \geq n_0$

Thus, $(f + g)(n) = f(n) + g(n) \leq (c + 1) f(n)$ for all $n \geq n_0$, which implies $f + g \in O(f)$.

On the other hand, for all n, $(f + g)(n) = f(n) + g(n) \geq f(n)$, so $(f + g) \in \Omega(f)$.

Therefore, since $f + g \in O(f)$ and $f + g \in \Omega(f)$, then $f + g \in \Theta(f)$.

2. Encodings

a.

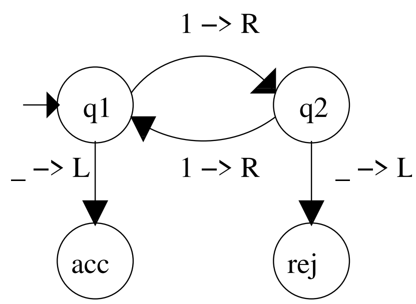
Type	Encoding	Length
Unary	1111111	7
Binary	111	3
Ternary	21	2
Hexadecimal	7	1

b. The algorithm takes n steps. The running time of the algorithm must be given in terms of l, the encoding length.

Type	Length	Running Time (n steps) as a function of l
Unary	$l = n$	$O(l)$
Binary	$l = \lfloor \log_2 n \rfloor + 1 > \log_2 n$	$O(2^l)$
Ternary	$l = \lfloor \log_3 n \rfloor + 1 > \log_3 n$	$O(3^l)$
Hexadecimal	$l = \lfloor \log_{16} n \rfloor + 1 > \log_{16} n$	$O(16^l)$

3. Turing machines, RAM programs and analysis of algorithms

a. Turing machine M_1:



Note: The symbol written to the tape is omitted, since it is the same already there.

Complexity of Turing machine M_1:

M_1 reads symbols until the first "_" is found, in which case it stops. Thus, the running time is $O(n)$.

b. The RAM program is as follows, encoding the end of string or empty space in the Turing machine by -1:

1. LOAD =1
2. STORE 1
3. READ ↑1 (at kth iteration $r_1 = k$ here)
4. STORE 2 ($r_2 := i_k$)
5. LOAD 1
6. ADD =2
7. STORE 1 (at kth iteration $r_1 = k + 2$ here)
8. LOAD 2

9. STORE ↑1 ($r_{k+2} := r_2$ (kth input value))
10. ADD 1
11. JZERO 16 (if $r_0 (= r_{k+1}) = -1$ go to 16)
12. LOAD 1
13. SUB =1
14. STORE 1 ($r_1 = k + 1$ here)
15. JUMP 3 (go to 3 for next iteration)
16. LOAD =3
17. STORE 1 ($r_1 := 3$ (the index of register holding the first input))
18. LOAD ↑1 (start simulation or $q_1, 1$)
19. SUB =1
20. JZERO 22 (check if symbol is 1)
21. JUMP 26 (if not go to q_1,⊔)
22. LOAD 1
23. ADD =1
24. STORE 1 (head move to the right)
25. JUMP 34 (transition to start of state q_2)
26. LOAD ↑1 (start simulation of q_1,⊔)
27. SUB =-1
28. JZERO 30 (check if symbol is -1)
29. JUMP 52 (if not, something is wrong; reject)
30. LOAD 1
31. ADD =-1
32. STORE 1 (head moves to the left)
33. JUMP 50 (transition to q_{accept})
34. LOAD ↑1 (start simulation of $(q_2, 1)$)
35. SUB =1
36. JZERO 37 (check if symbol is 1)
37. JUMP 42 (if not go to (q_2,⊔))
38. LOAD 1
39. ADD =1
40. STORE 1 (move head to right)
41. JUMP 18 (transition to begining of state q_1)
42. LOAD ↑1 (start simulation (q_2,⊔))
43. SUB =-1
44. JZERO 46 (check if symbol is -1)
45. JUMP 52 (if not, something wrong! reject)
46. LOAD 1

47. ADD = -1
48. STORE 1 (move head to the left)
49. JUMP 50 (transition to q_{accept})
50. LOAD =1 (start simulation of q_{accept})
51. HALT ($r_0 = 1$)
52. LOAD =0 (start simulation of q_{reject})
53. HALT ($r_0 = 0$)

Complexity of the RAM program:
The RAM program simulates the states of the Turing machine; for each state of the Turing machine, the RAM program takes at most 16 steps (steps 18-33 for q_1, 34-49 for q_2, 50-51 for q_{accept}, 52-53 for q_{reject}), so the simulation takes $O(n)$ steps. The initial loop that reads the n inputs also takes $O(n)$, so the total time is $O(n)$.

4. Multitape Turing machines

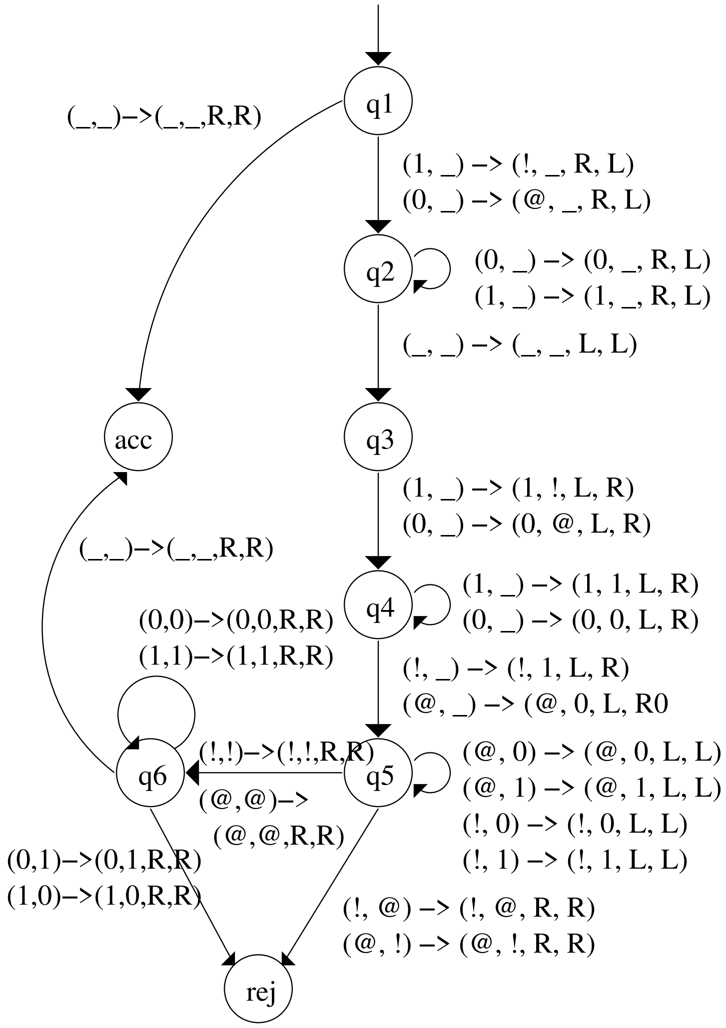
a. Idea for 2-tape machine:

1. Scan tape 1 all the way to the right, keeping tape 2 put, by moving it to the left. Note that, after reading the first symbol, it is substituted by another symbol: if the first symbol is 1, substitute it by ! and if it is 0, substitute it by @.

2. Scan tape 1 right to left while copying its symbols to tape 2 from left to right (mark first symbol of tape 2 too). See states q_3, q_4.

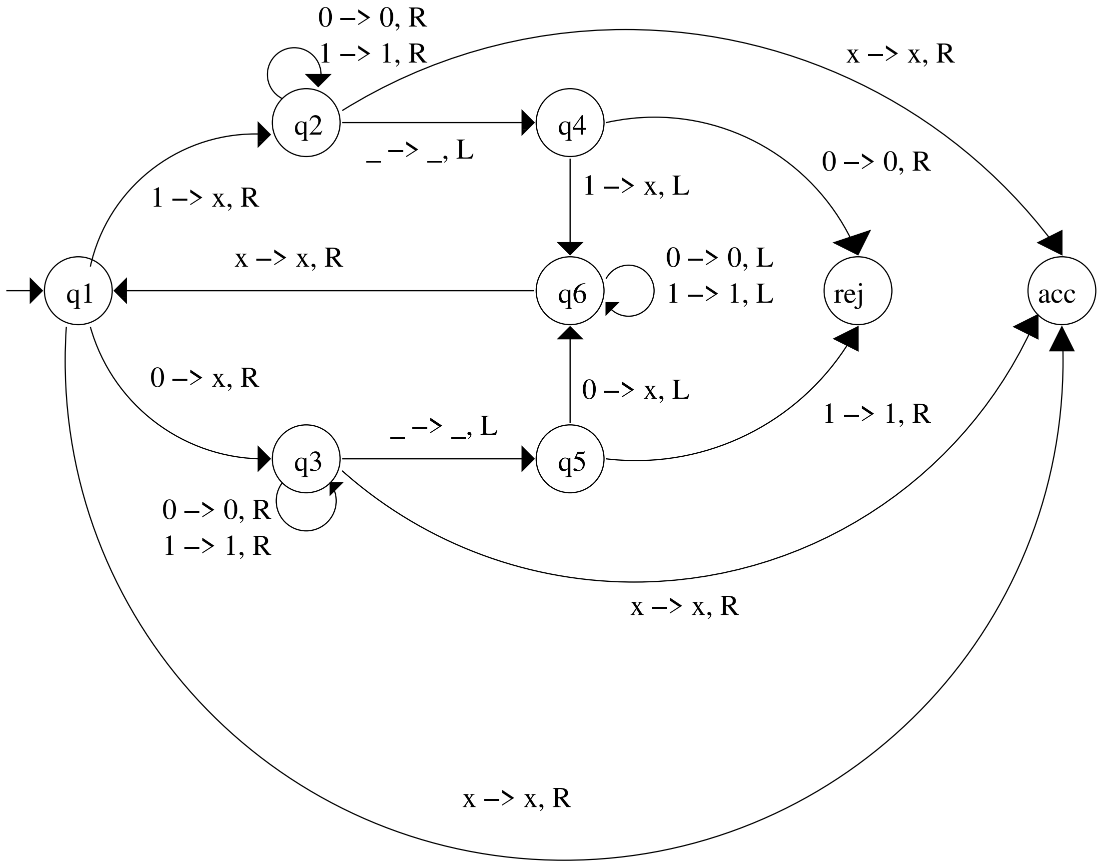
3. Move head of tape 2 to its beginning while leaving tape 1 put (by chosing to move left, while at the leftmost position). See state q_5.

4. Scan both tapes from left to right, checking that symbols in each tape are the same. If at any point the symbols differ, reject. If both tapes reach their end always with equal symbols, accept. See state q_6.



b. Idea for 1-tape machine:
Repeat the process:

- Read the current symbol on the tape. If it is an x, then accept. Otherwise, write x on top of it. Compare this symbol with the last one before _ or before an x. (If there are no more non-xs, accept.)
- If different, reject; otherwise, write x on top of the last symbol.
- Go left until an x is found. Move right.



5. P, NP and co-NP.

a. In all cases below since $L_1 \in \mathbf{P}$, $L_2 \in \mathbf{P}$, we know there exists a polynomial time algorithm A_1 that decides L_1 and A_2 that decides L_2. The proofs below show how to build A_3 which decides the appropriate language L_3 in polynomial time.

a1. $L_3 = L_1 \bigcup L_2$
Given x, A_3 will return 1 if and only if $A_1(x) = 1$ or $A_2(x) = 1$ (clearly polynomial time).

a2. $L_3 = L_1 \bigcap L_2$
Given x, A_3 will return 1 if and only if $A_1(x) = 1$ and $A_2(x) = 1$ (clearly polynomial time).

a3. $L_3 = L_1L_2$
A_3 `works as follows on string` x `of length` n.
`for each of the` $n+1$ `prefixes` x_0 `of` x `do`

```
        let x1 be such that x = x0x1
        if A1(x0) = 1 and A2(x1) = 1 then return 1
```

```
end for
return 0
```

A_3 runs in polynomial time since its complexity is in $\mathcal{O}(n(T_1(n) + T_2(n)))$ where $T_i(n)$ is the running time of algorithm A_i, which was assumed to be polynomial in n.

a4. $L_3 = \overline{L_1}$ (assuming L_1 *in* **P**)
Given x, A_3 will return 1 if and only if $A_1(x) = 0$ (clearly polynomial time).

b. Let $L \in \mathbf{P}$, we must show that $L \in$ **co-NP**. By part **a** we know that $\overline{L} \in \mathbf{P} \subseteq \mathbf{NP}$. By definition of **co-NP**, $\overline{\overline{L}} \in$ **co-NP**. But $\overline{\overline{L}} = L$. Therefore, $L \in$ **co-NP**.

c. This is equivalent to showing that **P**= **NP** implies **NP**=**co-NP**.
Assume **P**=**NP**.
Let $L \in$ **co-NP**. We know that $\overline{L} \in$ **NP**, but since **NP**=**P**, we conclude that $\overline{L} \in \mathbf{P}$. By part **a4** we know that $L = \overline{\overline{L}} \in \mathbf{P}$. Thus **co-NP** $\subseteq$ **P**. By part **b**, we know that **P** $\subseteq$ **co-NP**. Thus **P** = **co-NP**, and since **P** = **NP**, we conclude that **NP** = **co-NP**.

6. Decision problems vs. optimization problems
($\Rightarrow$)
Assume that LongestPathLength can be solved in polynomial time; let A be a polynomial time algorithm for LongestPathLength. We will provide an algorithm A' for deciding LongestPath.

```
Algorithm A':
    Input: < G, u, v, k >
        maxk ← A(G, u, v);
        if k > maxk then return 0 else return 1
```

Since A runs in polynomial time and A' simply calls A and does some extra constant number of steps, then A' runs in polynomial time.

($\Leftarrow$)

Assume that LongestPath can be decided in polynomial time; let A' be a polynomial time algorithm that decides LongestPath. We will provide an algorithm A that solves LongestPathLength.

```
Algorithm A:
    Input: < G, u, v >
    Output: The size of the longest path between u and v or −1
    if there is no path between them
        k ← n // Number of vertices in G
        while (k ≥ 0) and (A'(G, u, v, k) = 0) do
            k ← k − 1
        return k;
```

At most $n+2$ calls of A' are performed, plus a polynomial number of steps. Since A' runs in polynomial time, A runs in polynomial time.

7. Proving NP-completeness

Algorithm B decides ShortestPath (there are many algorithms possible):

Algorithm B:

Input: $< G, u, v, k >$

Output: 1/0, depending on whether there exists a simple path from u to v in G having at most k edges

- Let V be the vertex set and E be the edge set of G. Let $n = |V|$.
- Let M be an array on the vertex set.

```
Initialize M with 0 in all positions
M[u] ← 1
count ← 0
while ((count < k) and (count < n) and (M[v] = 0)) do
    for all (x ∈ V) do
        if (M[x] = 1) then
          for all (y ∈ V with {x, y} ∈ E) do
            M[y] = 1
          endfor
        endif
    endfor
    count ← count +1
endwhile
if (M[v] = 1) then return 1 else return 0
```

Correctness: It is easy to see that at the end of the while loop, all vertices at distance `count` of u are marked. We run the loop at most until `count` $= k$. Thus, the algorithm only returns 1 if v was reached within distance k, and correctly decides ShortestPath.

Polynomial time: The running time will depend on the data structure used to represent G. Assume an adjacency matrix is used to store G. Let $n = |V|$ and $m = |E|$.

- Initialization of M: $O(n)$
- `while` loop runs at most n times. Its embedded `for` loop runs n times, and the other embedded `for` loop runs at most n times. Thus, the `while` loop runs in $O(n^3)$.
- The final test runs in $O(1)$.

So the total running time is $O(n^3)$, which is polynomial in the length of the input.
Note: If one applies Djikstra's algorithm for ShortestPath, this algorithm could be done in $O(n\log n)$.

b. We show that LongestPath is NP-complete.
Step 1: LongestPath $\in$ NP

1. An algorithm $A(x, y)$ to verify LongestPath is described here.
   - The certificate y for the algorithm will be a sequence of vertices in the graph.
   - The verifier will check that the vertex sequence y satisfies the following conditions:
     - y has at least k vertices
     - y has no repeated vertices (since the path must be simple)
     - the first vertex of y is u and the last one is v
     - there is an edge connecting each pair of contiguous vertices in the sequence y

     The verifier returns 1 if and only if all of these conditions are satisfied.
2. This algorithm is a verifier:

- If $< G, u, v, k > \in$ LongestPath, then the longest simple path corresponds to a sequence of vertices y with no more than n vertices (so the certificate is short). For this y, $A(x, y) = 1$ since the longest path has length at least k.
- If $< G, u, v, k > \notin$ LongestPath, then there exists no vertex sequence corresponding to a simple path with length at least k. So for any y, one of the conditions checked by the algorithm will be violated, and this $A(x, y) = 0$.

3. The algorithm runs in polynomial time, since the conditions can easily be checked in polynomial time.

Step 2: We will show that
HamPath $\leq$p LongestPath

Step 3: We will describe a polynomial time algorithm F that computes a function $f : \{0, 1\}^* \to \{0, 1\}^*$ such that

$< G, u, v > \in$ HamPath if and only if $f(< G, u, v >) \in$ LongestPath

```
Algorithm F:
    Input:  < G, u, v >
        Let n be the number of vertices in G
        return < G, u, v, n >
```

Step 4: We now prove that $< G, u, v > \in$ HamPath if and only if $f(< G, u, v >) = < G, u, v, n > \in$ LongestPath.
Note that a hamiltonian path from u to v is a path passing through every vertex of G exactly once. So, a hamiltonian path from u to v is a simple path from u to v with n vertices (and vice versa).
If $< G, u, v > \in$ HamPath then there exists a hamiltonian path P from u to v with n vertices. So P is a simple path with n vertices, which implies $< G, u, v, n > \in$ LongestPath.
If $< G, u, v > \notin$ HamPath then there exists no hamiltonian path from u to v in G. So, every simple path from u to v in G has at most $n - 1$ vertices. So, $< G, u, v, n > \notin$ LongestPath.

Step 5: We show that algorithm F runs in polynomial time.

The algorithm does nothing but reading the input and outputting what was on the input and the number n which is smaller than the input size.
So the algorithm runs in linear time on the input size.