CSI 4105 Design and Analysis of Algorithms II — Fall 2001
Computer Science — University of Ottawa

Homework Assignment #1 (100 points, weight 10%)
Due: Thursday, October 18, at 10:00 a.m. (in lecture)

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1. (5 points) **Asymptotic notation review.**
Prove the following properties:

- **a.** If $g_1 \in O(f_1)$ and $g_2 \in O(f_2)$ then $g_1 + g_2 \in O(f_{max})$, where $f_{max}(n) = \max\{f_1(n), f_2(n)\}$.
- **b.** If $g_1 \in O(f_1)$ and $g_2 \in O(f_2)$ then $g_1 \cdot g_2 \in O(f_1 \cdot f_2)$.
- **c.** $g \in O(f)$ if and only if $f \in \Omega(g)$.
- **d.** $O(f) = O(g)$ if and only if $f \in O(g)$ and $g \in O(f)$.
- **e.** if $f \in O(g)$ and $g \in O(h)$ then $f \in O(h)$.

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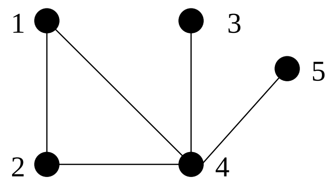
2. (15 points) **Graph representations review and graph encodings.**

Consider the graph:



(Review graph representations in section 22.1 of the textbook (23.1 in 1st edition))

- **a.** Draw its **adjacency matrix representation**.
  Using an alphabet containing all decimal digits and additional symbols of your choice (such as ; , ( )), give an encoding of the graph above as a string on the alphabet.
  Show that, for any graph with n vertices and m edges, the given encoding will produce a string with length polynomial in $n + m$,
- **b.** Do the same 3 steps above, for the **adjacency list representation**.

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3. (20 points) **Turing machines, RAM programs and analysis of algorithms.**
Consider the language:
$\textsc{Even} = \{x \in \{0,1\}^* : x \text{ is the binary representation of an even number }\}$.

- **a.** (5 points) Draw the state diagram for a Turing machine M_1 that decides $\textsc{Even}$. Analyse the complexity of M_1 by giving an asymptotic upper bound (big-Oh notation) for the number of steps taken by the machine on strings of length n.
- **b.** (8 points) Convert the Turing machine M_1 found on part **a.** into a RAM program, following the recipe given in the proof of Theorem 2 in pages 47-48 of the lecture notes. Similarly to part **a.**, analyse the complexity of the RAM program.
- **c.** (7 points) Give a constant time RAM program for deciding $\textsc{Even}$. Analyse its complexity, showing that its running time is in $O(1)$.

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4. (15 points) **P, NP and co-NP.**

We define the **complexity class co-NP** as: **co-NP** $= \{L \subseteq \{0,1\}^* :$ such that $\overline{L} \in$ **NP** $\}$.

a. (5 points) Prove that the class **P**, viewed as a set of languages, is closed under union, intersection, concatenation and complement. That is, prove that:

- if $L_1, L_2 \in \mathbf{P}$, then $L_1 \cup L_2 \in \mathbf{P}$;
- if $L_1, L_2 \in \mathbf{P}$, then $L_1 \cap L_2 \in \mathbf{P}$;
- if $L_1, L_2 \in \mathbf{P}$, then $L_1 L_2 \in \mathbf{P}$;
- if $L \in \mathbf{P}$, then $\overline{L} \in \mathbf{P}$;

b. (5 points) Prove that $\mathbf{P} \subseteq$ **co-NP**.

c. (5 points) Prove that if **NP** $\neq$ **co-NP** then **P** $\neq$ **NP**.

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5. (20 points) **Decision problems vs search problems.**

Show that if HAM-CYCLE $\in \mathbf{P}$, then the problem of listing the vertices of a hamiltonian cycle, in order, is polynomial-time solvable. Don't forget to prove that your algorithm runs in polynomial-time.
(Hint: use the decider for HAM-CYCLE as a subroutine for your algorithm.)

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6. (25 points) **Proving NP-completeness.**

The **subgraph-isomorphism problem** takes two undirected graphs G_1 and G_2 and asks whether G_1 is isomorphic to a subgraph of G_2. Show that the subgraph isomorphism problem is NP-complete.
(Hint: Reduce from CLIQUE.)

Recall the definitions:
We say that a graph $G' = (V', E')$ is a **subgraph** of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.
Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there exists a relabeling of the vertices of G_1 to vertices of G_2 maintaining the corresponding edges in G_1 and G_2, or more formally, if there exists a bijection $f : V_1 \to V_2$ such that $\{u, v\} \in E_1$ if and only if $\{f(u), f(v)\} \in E_2$.

IMPORTANT:
Marks will be alloted for conciseness and clarity.
You **must read and comply** with the policy on plagiarism stated in the course web page.