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Symmetries and sense of direction in labeled graphs☆

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Abstract

We consider edge-labeled graphs which model distributed systems, focus on properties of edge-labelings, and study their impact on graph classes. In particular, we investigate the relation between symmetries, topologies and sense of direction. We study symmetries based on the notion of view and of surrounding, and characterize the corresponding graph classes. Among other results, we show that the completely surrounding symmetric labeled graphs coincides with the class of Cayley graphs with Cayley labelings. We then focus on the relationship between symmetries and sense of direction in regular graphs. We characterize the class of regular labeled graphs with minimal symmetric sense of direction, as well as the class of those with group-based sense of direction. © 1998 Elsevier Science B.V. All rights reserved.

1. Introduction

A *distributed system* is a collection of autonomous entities which communicate by exchanging messages. The communication topology of the system is viewed as an edge-labeled undirected graph $G=(V,E)$ where nodes correspond to the system entities, edges represent pairs of neighboring entities (i.e., entities which can communicate directly), and each node $x \in V$ has a local label (usually called port number) $\lambda_x(\langle x,y \rangle)$ associated to each of its incident edges $\langle x,y \rangle$. The entire system is denoted by the labeled graph (G,λ) where $\lambda=\{\lambda_x\colon x\in V\}$. The properties of the labeling λ can be employed directly in the design of communication protocols for that system, so to

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yield more efficient distributed computations. In particular, it is well known that if λ satisfies the set of consistency constraints called *sense of direction* [8], the communication complexity of several distributed problem can be drastically reduced (e.g., see [7, 9, 12, 14, 16, 17, 21]).

In spite of its practical relevance and its theoretical interest, little is known on properties of edge-labeled graphs; actually, the study of edge-labelings and their impact on graph classes is a largely unexplored topic in the current research literature. In this paper we investigate several questions related to symmetries and sense of direction in edge-labeled graphs.

We consider symmetries based on two notions, *view* and *surrounding*, arising from the study of computability in *anonymous* distributed systems, that is where the entities do not have distinct names nor global identifiers. Views have been introduced in [24, 25] and extensively studied, sometimes with different names, e.g., [1, 2, 10, 13, 18–20, 22]; surroundings have been introduced and investigated in [10].

In the case of views, we consider the class of graphs which are (completely) view-symmetric, i.e. where all the nodes have the same view. A necessary and sufficient condition for the *existence* of a labeling λ which would make a graph G view-symmetric had been established in [25]. We find a reformulation of view-symmetry solely in terms of symmetry of the labeling and provide, in Section 3, a necessary and sufficient condition for a label graph (G,λ) to be view-symmetric. This result gives a complete characterization of the class of view-symmetric labeled graphs, and it leads to a simple and *optimal* algorithm for testing if in a labeled graph (G,λ) all the views are indistinguishable.

In Section 4, we consider the stronger form of symmetry based on the notion of surrounding. A labeled graph is k surrounding-symmetric (or S_k-symmetric) when there is a partition of the nodes in k classes, such that all the nodes in each class have the same surroundings; it is *completely* surrounding-symmetric when all the nodes have the same surroundings (i.e., $k=1$), and it is surrounding *asymmetric* if it is not S_k-symmetric for any $k<n$ where n is the number of nodes. We first establish that the class of completely surrounding symmetric is exactly the class of Cayley graphs with Cayley labelings. We then provide a characterization of the class of S_k-symmetric labeled graphs for $k>1$; we also show that there are regular graphs which are surrounding asymmetric with *any* labeling.

In Section 5, we consider sense of direction, study the link between symmetries and minimal sense of direction in regular graphs, and between sense of direction and a particular class of labelings based on commutative groups.

Informally, a system (G,λ) has *sense of direction* if it is possible to understand, from the labels associated to the edges, whether different walks from any given node x end in the same node or in different ones. A sense of direction is *minimal* if the labeling uses only $d(G)$ labels, where $d(G)$ is the maximum node degree in G. It has been shown in [6] that minimal sense of direction exists only in graphs which are *cycle symmetric*, i.e., informally, all nodes belong to the same number of cycles of the same lengths. Little is known in the general case.

We find an interesting and unsuspected link between minimal sense of direction in regular graphs and completely surrounding symmetric graphs. In fact, we prove that a regular graph has a minimal symmetrical sense of direction if and only if it is completely surrounding symmetric, i.e. if and only if it is a Cayley graph. This result provides a new characterization of Cayley graphs; in lieu of the (low) polynomial algorithm of [4] for testing for (weak) sense of direction, it gives a low polynomial algorithm to test if a labeled graph is indeed a Cayley graph with a Cayley labeling. No better technique is currently known. An equivalent result, for directed graphs, has been independently discovered by [5] using a different technique.

Finally, we study the relationship between sense of direction and the group-based labelings (CG-labelings) introduced in [23]. We give a negative answer to the "completeness" question, posed in [23], of whether the class of CG-labeled graphs coincides with the class of labeled graphs with sense of direction. (A positive answer would have implied a simpler definition of sense of direction). We actually show that the answer is negative even when the question is restricted to graphs with *anti-symmetric* labelings. Further, we show that the class of graphs with *uniform* CG-labelings [23] is a proper subset of the class of labeled graphs with minimal sense of direction; thus, the answer to the completeness question is negative even when restricted to graphs with *minimal* labelings.

The paper is organized as follows. In the next section we discuss the framework and some basic properties. In Section 3, we introduce the notion of view and we characterize the class of completely symmetric graphs. In Section 4, we introduce the notion of surrounding and characterize the classes of surrounding symmetric graphs. In Section 5, we investigate the relationship between sense of direction, completely surrounding symmetric graphs, and group-based labelings.

2. Framework

Let $G=(V,E)$ be a simple undirected graph; let $E(x)$ denote the set of edges incident to node $x \in V$, and $d(x)=|E(x)|$ the degree of x.

Given $G=(V,E)$ and a set Σ of labels, a *local labeling function* (or *local orientation*) of $x \in V$ is any injective function $\lambda_x : E(x) \to \Sigma$ which associates a distinct label $l \in \Sigma$ to each edge $e \in E(x)$. A set $\lambda = \{\lambda_x : x \in V\}$ of local labeling functions will be called a *labeling* of G, and by (G,λ) we shall denote the corresponding *(edge-)labeled graph*.

A labeling λ is *minimal* if it uses $d(G)=max\{d(x) \colon x \in V\}$ labels. It is *symmetric* if there exists a bijection $\psi : \Sigma \to \Sigma$ such that for each $\langle x,y\rangle \in E$, $\lambda_y(\langle y,x\rangle)=\psi(\lambda_x(\langle x,y\rangle))$; ψ will be called the edge-symmetry function.

Given two labeled graphs $(G=(V,E),\lambda)$ and $(G'=(V',E'),\lambda')$, a bijective function $\chi : V \to V'$ is a *labeled graph isomorphism* (or lg-isomorphism) between G and G' iff: 1) $\langle u,v\rangle \in E \Leftrightarrow \langle \chi(u),\chi(v)\rangle \in E'$; and 2) $\lambda(\langle u,v\rangle)=\lambda'(\chi(u),\chi(v)\rangle)$.

A *walk* π in G is a sequence of edges in which the endpoint of one edge is the starting point of the next edge. Let $P[x]$ denote the set of all the walks starting from

$x \in V$, $P[x,y]$ the set of walks starting from $x \in V$ and ending in $y \in V$. Let $\Lambda_x : P[x] \to \Sigma^+$ and $\Lambda = \{\Lambda_x : x \in V\}$ denote the extension of λ_x and λ, respectively, from edges to walks; let $\Lambda[x] = \{\Lambda_x(\pi) : \pi \in P[x]\}$, and $\Lambda[x,y] = \{\Lambda_x(\pi) : \pi \in P[x,y]\}$.

Given a walk $\pi = \langle e_1, e_2, \dots, e_k \rangle \in P[x,y]$ we will denote by $\overline{\pi}$ the reverse walk $\langle e_k, e_{k-1}, \dots, e_1 \rangle \in P[y,x]$; if $\Lambda_x(\pi) = \alpha$, we shall denote by $r_x(\alpha)$ the reverse string $\Lambda_y(\overline{\pi})$.

Given an edge symmetry function ψ, we shall denote by $\Psi : \Sigma^+ \to \Sigma^+$ its extension to strings; i.e., for $\alpha = a_1 \cdot a_2 \cdot \dots \cdot a_p \in \Sigma^+$, $\Psi(\alpha) = \psi(a_p) \cdot \dots \cdot \psi(a_1)$, where $\cdot$ denotes the concatenation operator.

Let $\rightharpoonup$ be the partial function $V \times \Sigma^* \to V$ such that (in infix notation) $u = u \rightharpoonup \varepsilon$, where ε is the empty string, and, for $\alpha \neq \varepsilon$, $v = u \rightharpoonup \alpha \Leftrightarrow \exists \pi \in P[u,v] \wedge \Lambda_u(\pi) = \alpha$. Let $u \rightharpoonup \alpha$ be defined; then, we shall denote by $[\![\alpha]\!]_u$ the set of strings $\Lambda[u, u \rightharpoonup \alpha]$.

3. Views and V-symmetries

A crucial concept when computing on anonymous networks is the one of *view*, introduced in [25]. The *view* $T_{(G,\lambda)}(v)$ of a node v in a labeled graph (G,λ) is an infinite, labeled, rooted tree "downward locally isomorphic" to G; i.e., such that there exists a mapping from the vertices of the tree to the vertices of G which maps the root of the tree to v, the children of the root to the neighbors of v and, recursively, the children of a node to the neighbors of that node. When no ambiguity arises, we shall denote a view $T_{(G,\lambda)}(v)$ simply by $T(v)$. For any integer $i \geqslant 0$, let $T^i(v)$ denote the i-view of node v, i.e., $T(v)$ truncated to distance i, where distance is defined in terms of edges. A labeled graph and its 2-view from node u are shown in Fig. 1.

It has been shown that in anonymous distributed systems (i.e., without node identities) the view of an entity represents the maximum information the entity can obtain by message transmission [25]. Furthermore, what is computable in such systems, depends on the multiplicity of the views [22, 25]. Intuitively, the least powerful systems are those where all nodes have the same view.

Definition 1 (*V-symmetry*). A labeled graph (G,λ) is *completely view symmetric* (or V-symmetric) when all nodes have the same view.

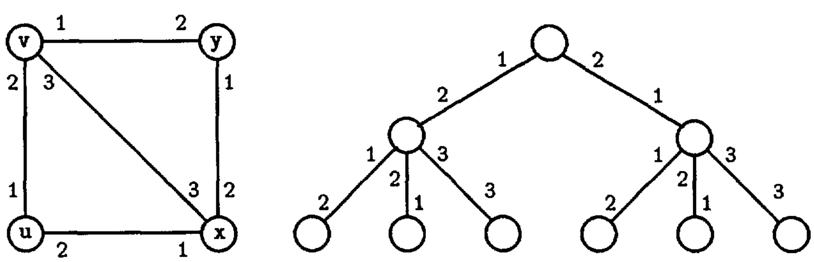

Fig. 1. A labeled graph and its 2-view $T^2(u)$ from node u.

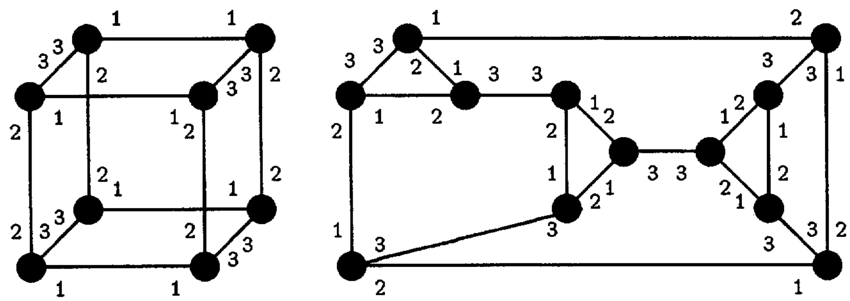

Fig. 2. V-symmetric graphs: (a) an hypercube (b) a minimum identity graph.

The first important question is to determine which labeled graphs are completely view symmetric. Obviously, V-symmetry can only exist in regular graphs.

The following *existential* result is due to [25]:

Theorem 1. *There exists a V-symmetric labeling of G iff G is regular and $\{1,2\}$-factorable.*

where a $p-factor$ of G is a spanning p-regular subgraph of G, and G is $\{1,2\}$-factorable if there is a set of 1- or 2-factors $F_1,\ldots,F_k$ $(F_i=(V,E_i))$ such that $E_1,\ldots,E_k$ constitute a partition of E.

It is possible, however, to derive a complete characterization of *V-symmetry* solely in terms of symmetry of the labeling, as stated in the next theorem:

Theorem 2. *A labeled graph (G,λ) is V-symmetric if and only if G is regular and λ is both minimal and symmetric.*

Proof. It follows immediately, by induction, from the recursive definition of view, d-regularity of G, and minimality and symmetry of λ. □

This characterization leads to a time-optimal algorithm for *V-symmetry testing*. In fact, from Theorem 2 it follows that, given a labeled regular graph (G,λ), to test if all nodes have the same view it is sufficient to test if the labeling is minimal and symmetric, which can easily be accomplished in time linear in the number of edges.

Two examples of V-symmetric graphs are shown in Fig. 2. In graph (a) the labeling is symmetric (the edge symmetry function is the identity function), and uses three labels; also in graph (b) the labeling is symmetric (with edge symmetry function $\psi(1)=2$, $\psi(2)=1$ and $\psi(3)=3$) and uses three labels. By Theorem 2, these two labeled graphs are V-symmetric.

From Theorems 1 and 2 it immediately follows a necessary and sufficient condition for the existence of a labeling which is simultaneously minimal and symmetric:

Theorem 3. *A regular graph G has a minimal symmetric labeling iff it is $\{1,2\}$-factorable.*

4. Surroundings and S-symmetries

4.1. Surroundings

In the previous section we considered the notion of view of a node in a labeled graph; a stronger notion is the one of *surrounding*, introduced in [10].

The *surrounding* $N_{(G,\lambda)}(u)$ of a node u in (G,λ) is the labeled graph lg-isomorphic to G, where the lg-isomorphism χ_u maps each node $v \in V$ to the set of strings $\Lambda[u,v]$ and u to the set of strings $\Lambda[u,u] \cup \{\varepsilon\}$, where ε is the empty string. When no ambiguity arises, we shall denote a surrounding $N_{(G,\lambda)}(u)$ by $N(u)$. As an example, the surrounding of node u in the labeled graph of Fig. 1 is shown in Fig. 3; notice the difference between surrounding and view.

It has been shown that, in anonymous distributed systems with sense of direction (a concept which will be discussed in Section 5), the surrounding of a node represents the maximum information that an entity can obtain by message transmissions [10]. Furthermore, what is computable in such systems, depends on the number of distinct surroundings as well as on their multiplicity (i.e., how many nodes have a given surrounding) [10].

Definition 2 (*S_k-symmetry*). A labeled graph (G,λ) is *S_k-symmetric* when there are k classes of nodes such that two nodes have the same surrounding iff they are in the same class.

Before proceeding to analyze the S_k-symmetries, we introduce a useful property which follows directly from the definition of surrounding:

Lemma 1. *There is an edge in $N(u)$ from node X to node Y labeled l, iff $\exists \alpha \in X$, $\beta \in Y$, such that $\beta = \alpha \cdot l$.*

4.2. S_1-symmetries

In this section we fully characterize the class of S_1-symmetric labeled graphs, i.e., when all nodes have the same surrounding. We show that it coincides with the class of Cayley graphs with Cayley labeling.

Given a set of generators Ω for a finite group Γ, a Cayley graph is a graph $G_\Gamma = (V,E)$, where the vertices correspond to the elements of the group ($V = \Gamma$) and the edges correspond to the action of the generators; that is $\langle x,y\rangle \in E$ iff $\exists g \in \Omega : x \circ g = y$, where $\circ$ is the operation of the group. The set of generators is closed under inverses; so we can consider the graph undirected.

Let $\Sigma = \Omega$; the natural labeling λ for a Cayley graph G_Γ is the following: $\forall \langle x,y\rangle \in E(x)$, $\lambda_x(\langle x,y\rangle) = g$, where g is the generator such that $y = x \circ g$. In the following, we shall call this labeling *Cayley labeling*.

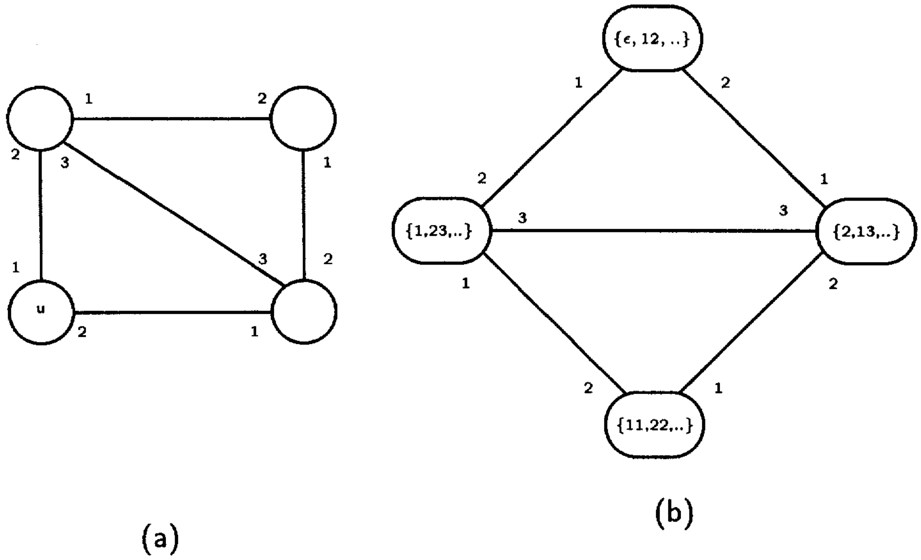

Fig. 3. A labeled graph and its surrounding from node u.

Theorem 4. *A labeled graph (G,λ) is S_1-symmetric iff G is a Cayley graph and λ is a Cayley labeling.*

Proof. ($\Rightarrow$) S_1-symmetry implies V-symmetry; hence, by Theorem 2, the graph is d-regular, the labeling is symmetric and uses d labels. Therefore, for any node v and any non-empty string $\beta \in \Sigma^+$, $\beta \in \Lambda[v]$. Furthermore, S_1-symmetry implies $[\![\alpha]\!]_u = [\![\alpha]\!]_v$ for each u and v and any $\alpha \in \Sigma^*$; thus, in the following, we shall use the notation $[\![\alpha]\!]$ without subscripts. Let $\Gamma = \{[\![\alpha]\!]\colon \alpha \in \Sigma^*\}$ and let $\odot : \Gamma \times \Gamma \to \Gamma$ be such that $[\![\alpha]\!] \odot [\![\beta]\!] = [\![\alpha \cdot \beta]\!]$. We now show that $(\Gamma, \odot)$ is a group. First, we prove that $\odot$ is associative: in fact, we have that $([\![\alpha]\!] \odot [\![\beta]\!]) \odot [\![\gamma]\!] = [\![\alpha \cdot \beta]\!] \odot [\![\gamma]\!] = [\![(\alpha \cdot \beta) \cdot \gamma]\!] = [\![\alpha \cdot (\beta \cdot \gamma)]\!] = [\![\alpha]\!] \odot [\![\beta \cdot \gamma]\!] = [\![\alpha]\!] \odot ([\![\beta]\!] \odot [\![\gamma]\!])$. We now show that $[\![\varepsilon]\!]$ is the identity of Γ: in fact, for each $[\![\alpha]\!] \in \Gamma$, $[\![\alpha]\!] \odot [\![\varepsilon]\!] = [\![\alpha \cdot \varepsilon]\!] = [\![\alpha]\!] = [\![\varepsilon \cdot \alpha]\!] = [\![\varepsilon]\!] \odot [\![\alpha]\!]$. Let $[\![\alpha]\!]^{-1} = [\![\Psi(\alpha)]\!]$, then $[\![\alpha]\!] \odot [\![\alpha]\!]^{-1} = [\![\alpha \cdot \Psi(\alpha)]\!] = [\![\varepsilon]\!]$ since $u \rightharpoonup \alpha \cdot \Psi(\alpha) = u = u \rightharpoonup \varepsilon$.

Let $\Omega = \{[\![a]\!] : a \in \Sigma\}$. Ω is a set of generators for Γ. In fact, for any $[\![\beta]\!] \in \Gamma$, by definition there exists $\alpha \in \Sigma^*$ s.t. $[\![\alpha]\!] = [\![a_1]\!] \odot [\![a_2]\!] \odot \cdots \odot [\![a_{|\alpha|}]\!] = [\![\beta]\!]$ where $[\![a_i]\!] \in \Omega$. Moreover, if $l \in \Sigma$, then $\Psi(l) \in \Sigma$; that is, $[\![l]\!] \in \Omega$, then $[\![l]\!]^{-1} = [\![\psi(l)]\!] \in \Omega$. Finally, $[\![\varepsilon]\!] \notin \Omega$ because there are no self-loops. Thus, $N(u)$ is the Cayley graph of the group $(\Gamma, \odot)$ with generator Ω. Since $N(u)$ is lg-isomorphic to (G,λ), the theorem follows.

($\Leftarrow$) By definition of Cayley graphs it directly follows that, if (G,λ) a Cayley graph with a Cayley labeling, then (G,λ) is completely surrounding symmetric. $\square$

We now examine the relationship between S_1-symmetry and a class of labelings discussed in [15]. Given a d-regular graph G, a *coloring and orientation* is a symmetric labeling with local orientation that uses d labels. A coloring and orientation is *regular* if, for any two walks $\pi_1 \in P[x]$ and $\pi_2 \in P[y]$ with $\Lambda_x(\pi_1) = \Lambda_y(\pi_2)$, $\pi_1 \in P[x,x]$ iff $\pi_2 \in P[y,y]$.

The following theorem is proved in [15].

Theorem 5. *A d-regular graph G has a regular coloring and orientation λ iff G is a Cayley graph on d generators and λ is the Cayley labeling.*

Thus, the notions of S_1-symmetry and of regular coloring and orientation coincide. This leads to the following alternative characterization of S_1-symmetric graphs:

Theorem 6. *A labeled graph (G, λ) is S_1-symmetric iff $\forall x, y \in V$, $\alpha \in \Sigma^*: x = x \rightarrow \alpha$ iff $y = y \rightarrow \alpha$.*

Proof. By Theorems 4 and 5 and by definition of regular coloring and orientation. □

4.3. S_k-symmetries

In this section, we give a necessary and sufficient condition for two nodes to have the same surrounding. Using this result, we then characterize the class of S_k-symmetric graphs; that is the graphs in which there are k different surroundings.

The following theorem gives a necessary and sufficient condition for two nodes to have the same surrounding.

Theorem 7. *For all $u, v \in V$ the following two conditions are equivalent:*
1. $\forall \alpha, \beta \in \Sigma^*$:
   (a) *$x = u \rightarrow \alpha$ is defined if and only if $y = v \rightarrow \alpha$ is defined, and*
   (b) $x = x \rightarrow \beta \Leftrightarrow y = y \rightarrow \beta$.
2. $N(u) = N(v)$.

Proof. ($1 \Rightarrow 2$): We will first show that for each node X in $N(u)$ there exists a node Y in $N(v)$ such that $X = Y$, and vice versa. Given a node X in the surrounding $N(u)$, let $\omega \in X$; this implies that $u' = u \rightarrow \omega$ is defined. By hypothesis (1a), also $v' = v \rightarrow \omega$ is defined; that is, there exists a node Y in the surrounding $N(v)$ such that $\omega \in Y$. By definition of string reverse, $v = v \rightarrow \omega \cdot r_v(\omega)$; by hypothesis 1 with $\alpha = \varepsilon$ and $\beta = \omega \cdot r_v(\omega)$, it follows that $u = u \rightarrow \omega \cdot r_v(\omega)$ and, since $u' = u \rightarrow \omega$, $u = u' \rightarrow r_v(\omega)$. Consider now an arbitrary $\gamma \in X$; we will show that $\gamma \in Y$. Since $\gamma \in X$, $u' = u \rightarrow \gamma$ and, since as shown above $u = u' \rightarrow r_v(\omega)$, it follows that $u' = u' \rightarrow r_v(\omega) \cdot \gamma$. By hypothesis 1 with $\alpha = \omega$ and $\beta = r_v(\omega) \cdot \gamma$, it follows that $v' = v' \rightarrow r_v(\omega) \cdot \gamma$; that implies $v' = v \rightarrow \omega \cdot r_v(\omega) \cdot \gamma$. The last expression becomes $v \rightarrow \gamma = v'$ since, by definition of reverse, $v = v \rightarrow \omega \cdot r_v(\omega)$; thus, $\gamma \in Y$. Since γ is arbitrary, it follows $X \subseteq Y$. Analogously, it can be shown that $Y \subseteq X$; thus, $X = Y$. That is, for every node in $N(u)$ there is one in $N(v)$

with the same set of strings. The vice versa can be shown with a similar reasoning. Hence, $N(u)$ and $N(v)$ have the same set of nodes. We now show that $N(u)$ and $N(v)$ have also the same set of labeled edges. Let X_1, X_2 be nodes in $N(u)$, and Y_1, Y_2 nodes in $N(v)$ such that $X_1 = Y_1$ and $X_2 = Y_2$. By Lemma 1, there is a (labeled) edge in $N(u)$ between X_1 and X_2 iff there is an edge with the same label between Y_1 and Y_2. Thus, $N(u) = N(v)$.

$(2 \Rightarrow 1)$: By contradiction, (Case a$'$) $\exists \alpha \in \Sigma^*$ such that $u \rightarrow \alpha$ is defined but $v \rightarrow \alpha$ is not, or (Case b$'$) $\exists \alpha, \beta \in \Sigma^*$ such that $u \rightarrow \alpha \cdot \beta = u \rightarrow \alpha$ but $v \rightarrow \alpha \cdot \beta \neq v \rightarrow \alpha$.

Case a$'$: By definition of surrounding $u \rightarrow \alpha$ being defined implies that there is a node X in $N(u)$ such that $\alpha \in X$. Since $N(u) = N(v)$, there is a node Y in $N(v)$ such that $X = Y$. Thus, $\alpha \in Y$; that is, $v \rightarrow \alpha$ is defined; a contradiction.

Case b$'$: Let X and Y be the two nodes in $N(u)$ and $N(v)$, respectively, such that $X = Y$, and let $\alpha \in X$. Since $u \rightarrow \alpha \cdot \beta = u \rightarrow \alpha$, then, $\alpha \cdot \beta \in X$ and, hence, $\alpha \cdot \beta \in Y$; this implies $v \rightarrow \alpha \cdot \beta = v \rightarrow \alpha$ which is a contradiction. □

We can now characterize the class of S_k-symmetric graphs.

Theorem 8. *A labeled graph (G, λ) is S_k symmetric iff there exists a partition $P = (P_1, \ldots, P_k)$ of the nodes such that $\forall i$, $\forall x, y \in P_i$, $\forall \delta \in \Sigma^*$:*
(a) $x \rightarrow \delta \in P_j$ iff $y \rightarrow \delta \in P_j$; and
(b) $x = x \rightarrow \delta$ iff $y = y \rightarrow \delta$.

Proof. $(\Rightarrow)$: Let $N_1, \ldots, N_k$ be an arbitrary order of the k different surroundings, and let $P = (P_1, \ldots, P_k)$ be a partition of V such that $P_i = \{u\colon N(u) = N_i\}$. By contradiction, suppose that (a) or (b) does not hold; that is, $\exists i, j \in \{1, \ldots, k\}$, $x, y \in P_i$, $\delta \in \Sigma^*$ such that: (a$'$) $x \rightarrow \delta \in P_j$ but $y \rightarrow \delta \notin P_j$, or (b$'$) $x = x \rightarrow \delta$ but $y \neq y \rightarrow \delta$.

Case a$'$: By construction, $N(x \rightarrow \delta) \neq N(y \rightarrow \delta)$; this implies the existence of β such that $[\![\beta]\!]_{x \rightarrow \delta} \neq [\![\beta]\!]_{y \rightarrow \delta}$. Thus, $[\![\delta \cdot \beta]\!]_x \neq [\![\delta \cdot \beta]\!]_y$ that implies $N(x) \neq N(y)$: a contradiction.

Case b$'$: In this case, $\delta \in [\![\varepsilon]\!]_x$ but $\delta \notin [\![\varepsilon]\!]_y$; this implies $[\![\varepsilon]\!]_x \neq [\![\varepsilon]\!]_y$, and, thus, $N(x) \neq N(y)$: a contradiction.

$(\Leftarrow)$: Let $P = (P_1, \ldots, P_k)$ be a partition of the node for which conditions (a) and (b) hold. We will show that for each $u, v \in P_i$, $N(u) = N(v)$. In order to show this, we prove that: $\forall \alpha, \beta \in \Sigma$: (1) $u' = u \rightarrow \alpha$ is defined iff $v' = v \rightarrow \alpha$ is defined, and (2) $u' = u' \rightarrow \beta$ iff $v' = v' \rightarrow \beta$.

By hypothesis (a) with $\delta = \alpha$, $x = v$ and $y = v$, we have that $u \rightarrow \alpha \in P_j$ iff $v \rightarrow \alpha \in P_j$, which implies $u \rightarrow \alpha$ is defined iff $v \rightarrow \alpha$ is defined (this proves 1), and that $u', v' \in P_l$ for some l. Thus by hypothesis (b) with $\delta = \beta$, $x = u'$ and $y = v'$, we have that $u' = u' \rightarrow \beta$ iff $v' = v' \rightarrow \beta$, that proves (2). It now follows from Theorem 7 that $N(v) = N(u)$. □

In Fig. 2(b) there is an example of a V-symmetric graph (G, λ) which is not S_k-symmetric for any $k < n$ where n is the number of nodes. Notice that G (known as "minimum identity graph" [3]) cannot be S_k-symmetric for $k < n$ *regardless of the choice of the labeling* because there are no isomorphisms between vertices in G. We shall call any such graph *surrounding-asymmetric*. An interesting open question is the characterization of these graphs and their properties.

5. Symmetries and sense of direction

In this section, we introduce the definition of sense of direction, show the existing link between symmetries and minimal sense of direction in regular graphs, and discuss the relationship with the CG-labelings of a graph, a particular class of labelings based on commutative groups.

5.1. Sense of direction

Given a labeled graph (G, λ), the system is said to have *Sense of Direction* when it is possible to understand, from the labels associated to the edges, whether different walks from any given node x end in the same node or in different ones (see, e.g. [4, 8, 11]). More precisely, sense of direction involves the existence of a consistent coding and a consistent decoding function.

Given (G, λ), a *consistent coding function* (or, simply, *coding function*) $\mathbf{c}$ for λ is any function with domain Σ^+, such that walks originating from the same node are mapped to the same value (called *local name*) if and only if they end in the same node; that is, $\forall x, y, z \in V$, $\forall \pi_1 \in P[x, y]$, $\pi_2 \in P[x, z]$, $\mathbf{c}(\Lambda_x(\pi_1)) = \mathbf{c}(\Lambda_x(\pi_2)) \Leftrightarrow y = z$. A coding function is *homonymous* when $\forall x, y \in V$, $\pi_1 \in P[x, x]$, $\pi_2 \in P[y, y]$: $\mathbf{c}(\Lambda_x(\pi_1)) = \mathbf{c}(\Lambda_y(\pi_2))$; that is, an homonymous coding function associates the same value to all the cycles in the graph.

Definition 3 ($\mathcal{WSD}$ – *Weak sense of direction*). A labeled graph (G, λ) has *Weak Sense of Direction* $\mathbf{c}$ iff $\mathbf{c}$ is a coding function for λ. Alternatively, we shall say that $\mathbf{c}$ is a $\mathcal{WSD}$ in (G, λ).

Let $\mathbf{c}$ be a $\mathcal{WSD}$ in (G, λ); if $\mathbf{c}$ is homonymous, we say that (G, λ) has an homonymous $\mathcal{WSD}$.

Given a coding function $\mathbf{c}$, a *consistent decoding function* (or, simply, *decoding function*) $\mathbf{d}$ for $\mathbf{c}$ is any function such that $\forall \langle x, y \rangle \in E(x)$, $\pi \in P[y, z]$ $\mathbf{d}(\lambda_x(\langle x, y \rangle), \mathbf{c}(\Lambda_y(\pi))) = \mathbf{c}(\lambda_x(\langle x, y \rangle) \cdot \Lambda_y(\pi))$.

Definition 4 ($\mathcal{SD}$ – *Sense of direction*). A labeled graph (G, λ) has *Sense of Direction* $(\mathbf{c}, \mathbf{d})$ iff $\mathbf{c}$ is a $\mathcal{WSD}$ and $\mathbf{d}$ is a decoding function for $\mathbf{c}$. Alternatively, we shall say that $(\mathbf{c}, \mathbf{d})$ is a $\mathcal{SD}$ in (G, λ).

Notice that $\mathcal{SD}$ is a stronger notion than $\mathcal{WSD}$; in fact, there exist labeled graphs with $\mathcal{WSD}$ but without $\mathcal{SD}$ (see, e.g., [4]).

5.2. Minimal sense of direction

In this section we consider labeled graphs (G, λ) where λ is minimal; i.e., it uses $d(G)$ labels. In particular, we focus on regular graphs. A labeled graph with minimal labeling and (weak) sense of direction is said to have *minimal (weak) sense of direction.*

An interesting problem is the characterization of the class of *regular* graphs with minimal sense of direction. In [6] it has been shown that a regular graph can have a minimal sense of direction only if is *cycle symmetric*, i.e., informally, all nodes belong to the same number of cycles of the same lengths. A complete characterization of regular graphs which have minimal (weak) sense of direction has not been found yet. In this section we move one step in this direction and completely characterize the class of regular graphs with minimal $\mathscr{S}\mathscr{D}$ when the labeling is *symmetric.*

Consider first the following lemma, established in [10].

Lemma 2. *Let (G, λ) have weak sense of direction. (G, λ) is S_1-symmetric iff it is V-symmetric.*

We can now prove the following equivalences.

Theorem 9. *Let (G, λ) be a regular graph with minimal symmetric labeling. The following statements are equivalent:*
(1) (G, λ) has $\mathscr{W}\mathscr{S}\mathscr{D}$
(2) (G, λ) has $\mathscr{S}\mathscr{D}$
(3) (G, λ) is a Cayley graph with Cayley labeling.

Proof. $(1) \Rightarrow (3)$: Let $\mathbf{c}$ be a consistent coding function for (G, λ). Since λ uses d labels, by Theorem 2, (G, λ) is V-symmetric. By Lemma 2, (G, λ) is S_1-symmetric. By Theorem 4, (G, λ) is a Cayley graph with Cayley labeling.

$(3) \Rightarrow (2)$: In [6], it has been proved that every Cayley graph has a minimal sense of direction with symmetric labeling.

$(2) \Rightarrow (1)$: It trivially follows by definition of sense of direction. $\square$

In spite of the simplicity of its proof, this result has some strong consequences. First of all, it uncovers an interesting and unsuspected link between minimal sense of direction and Cayley graphs, via symmetries of the surrounding. This provides a new characterization of Cayley graphs (with Cayley labelings) in terms of *consistency* of the sequence of labels. As a consequence, to *test* if a regular graph G with minimal symmetric labeling is a Cayley graph with a Cayley labeling is equivalent to test for the existence of a consistent coding function. Note that there is no need to test for the existence of a consistent *decoding* function because of the equivalence of the first two statements in the theorem.

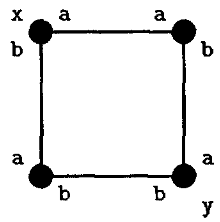

Fig. 4. A non-Cayley labeling of C_4 with minimum number of labels.

In lieu of the (low) polynomial algorithm of [4] for testing for (weak) sense of direction, this results implies the existence of a low polynomial algorithm for test if a labeled graph is Cayley. No better technique is currently known.

Let us show that the requirement in Theorem 9 that λ be symmetric is necessary; that is, without symmetry of the labeling, not all regular labeled graphs with minimal sense of direction are Cayley graph with Cayley labeling. Consider, for example, the labeled graph (G,δ) shown in Fig. 4. It is easy to verify that (G,δ) has a sense of direction. At the same time, G is a Cayley graph but δ is not a Cayley labeling: for node x, $a{\cdot}a=I$ but, for node y, $a{\cdot}a\neq I$, where I is the identity of the group.

Furthermore, in regular graphs with minimal sense of direction, *symmetry of the labels* and *homonymy of the coding function* coincide. In fact, we have that:

Theorem 10. *Let (G,λ) be a regular labeled graph with minimal sense of direction* $\mathbf{c}$. *λ is symmetric iff $\mathbf{c}$ is homonymous.*

Proof. Let (G,λ) be a d-regular labeled graph with minimal sense of direction.

($\Rightarrow$): Let ψ be the symmetry function. Let $\pi_1\in P[x,x]$ and $\pi_2\in P[y,y]$, with $x\neq y$. The graph is d-regular and uses d labels $(l_1,\dots,l_d)$, thus, by definition of consistent coding function, we have that: for every l_i, $1\leqslant i\leqslant d$, $\mathbf{c}(\Lambda_x(\pi_1))=\mathbf{c}(l_i\cdot\psi(l_i))=\mathbf{c}(\Lambda_y(\pi_2))$

($\Leftarrow$): Let $l_1,\dots,l_d$ be the d labels used. By contradiction, suppose λ is not symmetric. In this case, we would have two edges $\langle x,y\rangle$ and $\langle z,w\rangle$ such that $\lambda_x(\langle x,y\rangle)=a$, $\lambda_y(\langle y,x\rangle)=b$, $\lambda_z(\langle z,w\rangle)=a$, $\lambda_w(\langle w,z\rangle)=c$, with $c\neq b$.

By definition of homonymy, we have that $\mathbf{c}(\Lambda_x(\langle x,y\rangle\cdot\langle y,x\rangle))=\mathbf{c}(\Lambda_z(\langle z,w\rangle\cdot\langle w,z\rangle))$, that is: $\mathbf{c}(a,b)=\mathbf{c}(a,c)$.

Since the graph is d-regular, uses d labels, and $c\neq b$ there must exist an edge $\langle y,y'\rangle$, with $y'\neq x$ such that $\lambda_y(\langle y,y'\rangle)=c$. But, by definition of consistent coding function, we have that: $\mathbf{c}(\Lambda_x(\langle x,y\rangle\cdot\langle y,x\rangle))\neq\mathbf{c}(\Lambda_x(\langle x,y\rangle\cdot\langle y,y'\rangle))$, that is: $\mathbf{c}(a,b)\neq\mathbf{c}(a,c)$, yielding the contradiction. □

Thus, in Theorem 9, the assumption of symmetry of the labeling can be replaced by the assumption of homonymy of the coding function. The result of Theorem 9, with the assumption of homonymy, has been independently discovered by [5] and shown to hold also for directed graphs.

5.3. *Group sense of direction*

A particular labeling, here called *Commutative Group labeling*, or CG-labeling, has been defined in [23]; in this section, we describe the relations between sense of direction, CG-labelings, and Cayley graphs.

5.3.1. *CG-labeling*

Let Γ be a commutative group with binary operation $+$ and identity 0. A labeled graph $(G=(V,E),\lambda)$ has a *CG-labeling* (based on Γ) iff (1) there exists a bijection $\mathcal{N}: V \to \Gamma$, and (2) $\lambda_u(\langle u,v\rangle)+\mathcal{N}(u)=\mathcal{N}(v)$ [23].

Thus, given any graph G, it is always possible to construct a CG-labeling by choosing appropriate commutative groups. Observe that a CG-labeling satisfies the *anti-symmetry* property $\lambda_u(\langle u,v\rangle)=-\lambda_v(\langle v,u\rangle)$, and thus, it is a special case of symmetric labeling. For each $a_0 \cdot a_1 \cdot \cdots \cdot a_n = \alpha \in \Gamma^*$, let $\oplus : \Gamma^* \to \Gamma$ be such that $\oplus(\alpha)=a_0+a_1+\cdots+a_n$.

Any graph with CG-labeling has homonymous sense of direction, as shown in the next theorem.

Theorem 11. *A system (G,λ), where λ is a CG-labeling has homonymous sense of direction $(\oplus,\oplus)$.*

Proof. Let λ be a CG-labeling based on Γ in (G,λ). By Theorem 6 in [23], λ has the closed walk property, that is for each $\pi \in P[u,v]$, $\oplus(\Lambda_u(\pi))=0 \Leftrightarrow u=v$. By contradiction suppose that $\oplus$ has the closed walk property and it is not a consistent coding function for λ. This means that there exist three nodes x, y, z and two walk $\pi_1 \in P[x,y]$, $\pi_2 \in P[x,z]$ s.t. $\oplus(\Lambda_x(\pi_1))=\oplus(\Lambda_x(\pi_2)) \not\Leftrightarrow y=z$. Let π be the reverse walk $\overline{\pi_1}$ of π_1 concatenated with π_2, then $\oplus(\pi)=\oplus(\Lambda_y(\overline{\pi_1}\cdot\pi_2))=\oplus(\Lambda_y(\overline{\pi_1}))+\oplus(\Lambda_x(\pi_1))$. By extending the anti-symmetry property from edges to walks, $\oplus(\Lambda_y(\overline{\pi_1}))=-\oplus(\Lambda_x(\pi_1))$. By the closed walk property on π, $\oplus(\Lambda_y(\pi))=0 \Leftrightarrow y=z$, that implies $\oplus(\Lambda_x(\pi_1))=\oplus(\Lambda_x(\pi_2)) \Leftrightarrow y=z$ which is a contradiction. Analogously, it is easy to see that $\oplus$ is also a consistent decoding function; thus, $(\oplus,\oplus)$ is a sense of direction in (G,λ). Homonymy follows from the commutativity of the coding function. $\square$

In the following, we shall refer to such a sense of direction as a *CG-sense of direction.*

5.3.2. *CG-$\mathscr{SD}$ and $\mathscr{SD}$*

In [23], Tel posed the question of whether any sense of direction is a CG sense of direction; that is, if the definition of CG-labelings completely defines sense of direction. Observe that a positive answer to this question, which we shall call the "completeness" question, would have implied a simpler equivalent definition of sense of direction. In

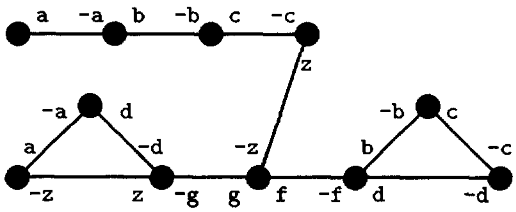

Fig. 5. A sense of direction that is not CG-sense of direction.

the following, we will settle this question with a negative answer, even for restricted classes of labelings.

To verify that there are senses of direction which are not CG-senses of direction, it suffices to show a labeled graph with sense of direction where the labeling is *not* anti-symmetric, or where *all* the coding functions are *not* homonymous (e.g., the graph of Fig. 4). However, even when considering only anti-symmetric labelings, there are homonymous sense of direction which are not CG-senses of direction, as shown by the following theorem.

Theorem 12. *The class of labeled graphs with CG-sense of direction is a proper subset of the class of anti-symmetrically labeled graphs with homonymous sense of direction.*

Proof. By Theorem 11, any labeled graph with CG-sense of direction has sense of direction. Consider now the labeled graph (G,λ) of Fig. 5. Suppose, by contradiction, that there is a CG-sense of direction. We must have that (1) $b+c=d$ (since $b\cdot c\cdot -d$ is a cycle); (2) $a+d=-z$ (since $a\cdot d\cdot z$ is a cycle); while, (3) $a+b+c+z\neq 0$ (since $a\cdot b\cdot c\cdot z$ is not a cycle). Substituting (1) in (3), we have that: $a+b+z\neq 0$, which contradicts (2). However, in this graph there exists an homonymous sense of direction that satisfies the anti-symmetric property. Consider the coding function $\mathbf{c}:\Sigma^+\to\Sigma^+$ such that $\forall\alpha=a_0\cdot a_1\cdot\cdots\cdot a_k\in\Sigma^+$, $\mathbf{c}(a_0\cdot a_1\cdot\cdots\cdot a_k)=a_0\ominus a_1\ominus\cdots\ominus a_k$, where $\ominus$ is a *noncommutative* operator such that $\forall x\in\Sigma$: $x\ominus\varepsilon=x$, $\varepsilon\ominus x=x$, $x\ominus -x=\varepsilon$, $-x\ominus x=\varepsilon$; moreover, $a\ominus d=-z$; $b\ominus c=d$. It is easy to verify that $\mathbf{c}$ is a consistent coding function in (G,λ); furthermore, $\mathbf{c}$ is homonymous since, by definition $x\ominus -x=\varepsilon$.

Furthermore, it is easy to see that $\mathbf{d}:\Sigma\times\Sigma^+\to\Sigma^+$, defined as follows, $\forall a\in\Sigma$, $\alpha\in\Sigma^+$, $\mathbf{d}(a,\mathbf{c}(\alpha))=a\ominus\mathbf{c}(\alpha)$ is a consistent decoding function for $\mathbf{c}$. Thus, $(\mathbf{c},\mathbf{d})$ is a sense of direction in (G,λ). □

In other words, the answer to the "completeness" question is negative even when restricted to *anti-symmetric* labelings and *homonymous* codings.

5.3.3. *Minimal $\mathscr{SD}$ and uniform CG-$\mathscr{SD}$*

We now consider the relationship between minimal sense of direction in regular graphs and a particular case of *CG*-labelings called *uniform.*

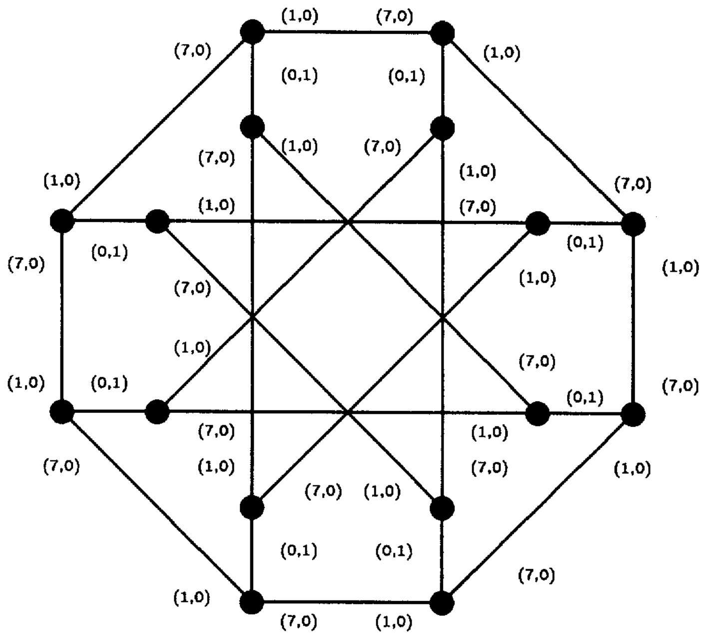

Fig. 6. A non-commutative Cayley graph.

A CG-sense of direction (based on Γ) is *uniform* if every node has the same collection of local labels Ω [23]. Note that, by definition, uniform sense of direction can exist only in regular graphs; in other words, "uniform" and "minimal" are synonyms. Further observe that any graph (G, λ) with a CG-sense of direction is a subgraph of a commutative Cayley graph with a Cayley labeling. Hence, if (G, λ) has uniform CG-sense of direction then G is regular and λ is a minimal and symmetric sense of direction; thus, (G, λ) is a Cayley graph with a Cayley labeling. In particular, a uniform CG-sense of direction based on Γ is a Cayley labeling of the commutative Cayley graph Γ with set of generators Ω.

Summarizing, the class of labeled graphs with uniform CG-sense of direction coincides with the class of *commutative* Cayley graphs with Cayley labeling. This implies that any Cayley graph of a non-commutative group has minimal sense of direction (e.g., the graph of Fig. 6), but, by definition, does not have uniform CG-sense of direction. That is, the class of regular graphs with uniform CG-sense of direction is a *proper subset* of the class of regular graphs with minimal sense of direction.

We can actually prove the stronger statement that there exist graphs with minimal $\mathcal{SD}$ which *cannot be relabeled* to be the Cayley graph of a commutative group, and thus to have uniform CG-sense of direction as will be shown in the next theorem.

Theorem 13. *The class of regular graphs G for which there exist a labeling λ such that (G, λ) has uniform CG-sense of direction is a proper subset of the class of regular* *graphs G for which there exist a labeling λ such that (G, λ) has minimal sense of direction.*

Proof. From Theorem 9, it follows that every labeled graph with uniform CG-sense of direction has also a minimal sense of direction. We now show that the converse is not true. Consider the graph G shown in Fig. 6. Consider also the labeling δ for G shown in Fig. 6. (G, δ) is a Cayley graph based on the group $\mathbf{Z}_8 \times \mathbf{Z}_2$ with *non-commutative* operation $+$ such that $(i,x)+(j,y)=(i+j\cdot 3^x \bmod 8, x+y \bmod 2)$ and set of generators $\{(1,0),(7,0),(0,1)\}$. By Theorem 9, (G, δ) has a minimal sense of direction.

Suppose, by contradiction, that there exists a labeling λ such that (G, λ) is the Cayley graph of a group with commutative operator $*$ and generators (i.e., labels) a, b, c. Without loss of generality, let $a*b=0$ and $c*c=0$. Then, by commutativity, $a*c*b*c=0$; on the other hand, this implies that there must exist a cycle of length 4 in G. A contradiction. Thus, any CG-labeling for G must use more than 3 labels. □

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