Levels of Sense of Direction in Distributed Systems

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Abstract

Sense of Direction refers to a set of consistency properties of distributed systems. The presence of such properties is known to have an impact on the communication complexity of several distributed problems. In this paper we consider levels of sense of direction (weak, standard, and symmetric). We study these levels of sense of direction with respect to a set of consistency properties (associativity, backward decoding, everywhere consistency, and name symmetry). In particular, we investigate the role played by associativity in standard sense of direction, and we study under what conditions a sense of direction is symmetric.

1 Introduction

A *distributed system* is a collection of autonomous entities communicating by the exchange of messages. The communication topology of the system can be represented as a graph $G(V, E)$ where nodes correspond to the system entities and edges represents pairs of neighbouring entities (i.e., entities which can communicate directly). Each entity has a local (partial) view of the system. In particular, a node has *local orientation*, i.e. it has a distinct label, sometimes called *port number*, associated to each of its incident edges. Let $\lambda_x(\langle x, z\rangle)$ be the label associated by $x \in V$ to the edge $\langle x, z\rangle \in E$. Note that, since labels are assigned locally, two neighbours may associate different labels to their incident edge; i.e., possibly $\lambda_x(\langle x, z\rangle) \neq \lambda_z(\langle z, x\rangle)$. Each node x refers to the other nodes using local labels called *names*; let $\beta_x(y)$ be the name associated by x to the node y. These local names are *not* necessarily identities (i.e., unique global identifiers); in fact, the system could be anonymous. The entire system can thus be described by the triple (G, λ, β), where $\lambda = \{\lambda_x : x \in V\}$ and $\beta = \{\beta_x : x \in V\}$.

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Some assignments of labels (i.e., some labelings λ) have properties which can be exploited in the solution of distributed problems. In particular, the set of consistency constraints called *Sense of Direction* [6] is well known to have a strong impact both on the communication complexity of many distributed problems, and on the computing power of distributed systems (e.g., [1, 9, 5, 7, 11, 12, 13, 14, 15]). Because of its impact on complexity, the characterization and analysis of sense of direction and its properties has been the object of intensive research (e.g., [3, 2, 4, 6, 9, 10]).

In this paper, we consider the (standard) sense of direction, as well as two other levels of global consistency constraints: a weak form, called *weak sense of direction* and a stronger form, called *symmetric sense of direction*, which exists only in systems with symmetric edge labelings. These other two levels are also of particular interest. Some properties of weak sense of direction have been identified and studied; it has been observed that, for most problems, it has the same power as sense of direction [9], and it is much less expensive to construct [3]. The power of symmetric sense of direction has not been deeply explored yet; it is interesting to note that almost all the senses of direction studied in the literature are actually symmetric ones.

We study these levels of sense of direction with respect to a set of consistency properties: *associativity*, *backward decoding*, *everywhere consistency*, and *name symmetry*. The paper is organized as follows. In the next section we introduce the basic definitions and the terminology. In Section 3, we focus on consistency and associativity, and study the relationship between associativity, weak sense of direction and standard sense of direction. In Section 4, we consider symmetric sense of direction and study under what conditions a sense of direction is symmetric. In particular, we focus on systems with edge-symmetry and investigate the function played by backward decoding.

2 The Framework

In this section, we use the notation, terminology and definitions of [6] with only few simplifications.

2.1 Notations and Definitions

Let $G = (V, E)$ be a connected simple graph describing the communication topology of a distributed system. Let $E(x)$ denote the set of edges incident to node x. Given a graph $G = (V, E)$ and a set $\mathcal{L}$ of labels, a *local edge-labeling* (or labeling) function for $x \in V$ is any function $\lambda_x : E(x) \to \mathcal{L}$ which associates a label $l \in \mathcal{L}$ to each of its incident edges $e \in E(x)$. If λ_x is injective (i.e., it associates distinct labels to different neighbours), it is said to be a *local orientation*, The *labeling* λ of G is the set of local labeling functions, that is $\lambda = \{\lambda_x : x \in V\}$.

In the following we shall denote by $P[x]$ the set of paths starting from node x, and by $P[x, y]$ the set of paths starting from x and terminating in node y. Let

Λ be the natural extension of the labeling function λ from edges to paths: given a labeling λ and a node $x \in V$, let $\Lambda_x : P[x] \to \mathcal{L}^+$ be the *path-labeling function* defined as follows: $\forall \pi \in P[x]$, $\Lambda_x(\pi) = (\lambda_x(\langle x, x_1 \rangle), \ldots, \lambda_{x_{m-1}}(\langle x_{m-1}, x_m \rangle))$ where $\pi = [\langle x, x_1 \rangle, \ldots, \langle x_{m-1}, x_m \rangle]$.

Each node x denotes to the other nodes using local names. Given a node $x \in V$ and a set $\mathcal{N} \subseteq \mathcal{L}^+$ of names, a local *node-labeling (or naming) function* for $x \in V$ is a function $\beta_x : V \to \mathcal{N}$ such that, $\forall y, z \in V$, $\beta_x(y) = \beta_x(z)$ iff $y = z$. The set of local names used by x to denote the other nodes is called the *local view* of x. Let us stress that these local names are *not* necessarily identities (i.e., unique global identifiers); in fact, the system could be anonymous. The *naming* function β for (G, λ) is the set of local naming functions, that is $\beta = \{\beta_x : x \in V\}$.

The entire system will be denoted by the triple (G, λ, β).

2.2 Sense of Direction

Consider a system (G, λ, β). Informally, in the system there is *weak sense of direction* if there is a coding function which allows to understand, from the edge labels of the paths, whether or not two paths starting from the same node end in different nodes.

Definition 1 Consistent Coding Function
A (consistent) coding function *is any function $f : \mathcal{L}^+ \to \mathcal{N}$ such that $\forall x, y \in V$, $\pi \in P[x, y]$, $f(\Lambda_x(\pi)) = \beta_x(y)$.*

As a consequence, we have: $\forall x, y, z \in V, \forall \pi_1 \in P[x, y], \pi_2 \in P[x, z], f(\Lambda_x(\pi_1)) = f(\Lambda_x(\pi_2)) \Leftrightarrow y = z$. In other words, paths originating from the same node are mapped to the same value if and only if they end in the same node.

Definition 2 $\mathcal{WSD}$ - Weak Sense of Direction.
A system (G, λ, β) has Weak Sense of Direction *($\mathcal{WSD}$) iff there exists a consistent coding function f.*

We shall also say that f *is a $\mathcal{WSD}$ in* (G, λ, β), or that (G, λ, β) has weak sense of direction f. In the following, when no ambiguity arises, we shall omit the reference to (G, λ, β) and the adjective "consistent" when referring to coding functions.

Informally, a system has (standard) sense of direction if, in addition to a coding function, there exists also a decoding function which allows a node to translate the local views of its neighbours.

Definition 3 Consistent Decoding Function
Given a coding function f, a (consistent) decoding function *for f is any function $h : \mathcal{L} \times \mathcal{N} \to \mathcal{N}$ such that $\forall \langle x, y \rangle \in E(x), \pi \in P[y, z]$,*
$h(\lambda_x(\langle x, y \rangle), f(\Lambda_y(\pi))) = \beta_x(z)$

The existence of a (consistent) decoding function is clearly a crucial property since it allows the nodes to solve global problems while working solely and truly in a local mode. In the following, we shall omit the adjective "consistent" when referring to a consistent decoding function.

Definition 4 $\mathcal{SD}$ - Sense of Direction
A system (G, λ, β) *has* Sense of Direction *($\mathcal{SD}$) iff it has weak sense of direction* f *and there exists a decoding function* h *for* f*.*

We shall also say that (f, h) *is a* $\mathcal{SD}$ *in* (G, λ, β), or that (G, λ, β) has sense of direction (f, h).

The *hierarchical relationship* between weak and standard sense of direction is *strict*: there are systems with weak sense of direction for which there is no decoding function. Actually, there are systems where *every* coding function has *no* decoding. Consider for example the graph of Figure 1, where every edge is given the same label by both end nodes; it has been shown in [3] that, in this system, no coding function can be consistently decoded.

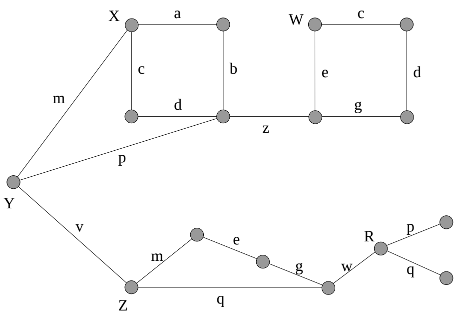

Figure 1: A system where every coding function is not decodable.

3 Consistency and Associativity

In this section, we consider the properties of *everywhere consistent* decoding and *(everywhere) associative* coding. We shall study the relationship between these properties and sense of direction.

3.1 Everywhere Consistent Decoding

In systems with (standard) sense of direction, the consistency required by a decoding function is limited to the sequences of labels (or *strings*) associated to paths; no requirements is made for strings which do not appear in the as paths in the graph. The requirement that this consistency must hold for all strings in $\mathcal{L}^+$ is called everywhere consistency.

Definition 5 Everywhere Consistent Decoding
Given a coding function f, a decoding function h for f is everywhere consistent *iff* $\forall a \in \mathcal{L}, \alpha \in \mathcal{L}^+$: $h(a, f(\alpha)) = f(a \cdot \alpha)$.

Clearly, everywhere consistency implies consistency. The converse is not necessarily true.

Any system with sense of direction has, by definition, a consistent decoding. This decoding function h may or may not be everywhere consistent for the coding f. However there is always a coding function (possibly different from f) for which h is everywhere consistent. In other words, every system with sense of direction has an everywhere consistent decoding.

Theorem 1 *If (f, h) is a $\mathcal{SD}$, then there exists a consistent coding function $\overline{f}$ such that $(\overline{f}, h)$ is a $\mathcal{SD}$ and h is everywhere consistent for $\overline{f}$.*

Proof Define the coding function $\overline{f}$ as follows: $\forall a \in \mathcal{L}$, $\overline{f}(a) = f(a)$, and $\forall w \in \mathcal{L}^+$:

$$\overline{f}(a \cdot w) = \begin{cases} f(a \cdot w) & \text{if } \exists x \in V, \pi \in P[x] \text{ s.t. } a \cdot w = \Lambda_x(\pi) \\ h(a, \overline{f}(w)) & \text{otherwise} \end{cases}$$

We first prove that $\overline{f}(w)$ is well defined by induction on the length of w. When $|w| = 1$, we have $\overline{f}(w) = f(w)$ which is defined. Let it hold for each w such that $|w| \leq n$. Consider the string $a \cdot w \in \mathcal{L}^+$ where $a \in \mathcal{L}$. If there exist $x \in V$ and $\pi \in P[x]$ s.t. $\Lambda_x(\pi) = a \cdot w$ then $\overline{f}(a \cdot w) = f(a \cdot w)$ which is defined. Otherwise, $\overline{f}(a \cdot w)$ is defined because $\overline{f}(a \cdot w) = h(a, \overline{f}(w))$ and $\overline{f}(w)$ is defined.
Note that f and $\overline{f}$ agree on paths: $\forall x \in V, \pi \in P[x]$ $f(\Lambda_x(\pi)) = \overline{f}(\Lambda_x(\pi))$.
Second, we prove that $\overline{f}$ is consistent. Since f and $\overline{f}$ agree on paths, $\forall x, y, z \in V$ and for each $\pi_1 \in P[x, y], \pi_2 \in P[x, z]$, we have $\overline{f}(\Lambda_x(\pi_1)) = \overline{f}(\Lambda_x(\pi_2))$ iff $f(\Lambda_x(\pi_1)) = f(\Lambda_x(\pi_1))$. Furthermore, by consistency of f, $f(\Lambda_x(\pi_1)) = f(\Lambda_x(\pi_1))$ iff $y = z$.
We now prove that h is a consistent decoding for $\overline{f}$. We have: $\forall \langle x, y\rangle \in E(x)$ and $\forall \pi \in P[y, z]$: $h(\lambda_x(\langle x, y\rangle), \overline{f}(\Lambda_y(\pi))) = h(\lambda_x(\langle x, y\rangle), f(\Lambda_y(\pi)))$, since $\overline{f}$ and f agree on paths. On the other hand, by consistency of h, it follows: $h(\lambda_x(\langle x, y\rangle), f(\Lambda_y(\pi))) = f(\lambda_x(\langle x, y\rangle) \cdot \Lambda_y(\pi))$. By the existence of the path $\langle x, y\rangle \cdot \pi$ and by definition of $\overline{f}$, we have $\overline{f}(\lambda_x(\langle x, y\rangle) \cdot \Lambda_y(\pi)) = f(\lambda_x(\langle x, y\rangle) \cdot \Lambda_y(\pi))$. Thus h is consistent and $(\overline{f}, h)$ is a $\mathcal{SD}$.
Finally we prove that h is everywhere consistent, that is $h(a, \overline{f}(w)) = \overline{f}(a \cdot w)$ for each $w \in \mathcal{L}^+$. If there exist $x, y, z \in V$ and $\langle x, y\rangle \in E(x)$ and $\pi \in P[y, z]$ s.t. $a = \lambda_x(\langle x, y\rangle)$ and $w = \Lambda_x(\pi)$ then, by consistency of h, $h(a, f(w)) = f(a \cdot w)$. Furthermore the existence of the paths $\langle x, y\rangle \cdot \pi$ and π implies $\overline{f}(a \cdot w) = f(a \cdot w)$ and $\overline{f}(w) = f(w)$. Thus $h(a, \overline{f}(w)) = h(a, f(w)) = f(a \cdot w) = \overline{f}(a \cdot w)$. If no x, y, z and path π exist $\overline{f}(a \cdot w) = h(a, \overline{f}(w))$ by definition of $\overline{f}$. □

3.2 Everywhere Associative Coding

In this section, we consider the properties of *associativity* and *everywhere associativity* of coding functions, and study their relationship with sense of direction.

Definition 6 Associative Coding
A coding function f *is* associative *iff:* $\forall \langle x, y\rangle \in E(x),\ \forall \pi \in P[y]$

$$f([\lambda_x(\langle x, y\rangle) \cdot f(\Lambda_y(\pi))]) = f([\lambda_x(\langle x, y\rangle) \cdot \Lambda_y(\pi)])$$

By extending the associative requirements from strings corresponding to sequences of labels on paths to all strings, we have the stronger property of associativity everywhere.

Definition 7 Everywhere Associative Coding
A coding function f *is* everywhere associative *iff:* $\forall a \in \mathcal{L},\ \forall w \in \mathcal{L}^+$

$$f(a \cdot w) = f(a \cdot f(w))$$

What is the relationship between associativity and sense of direction? It is known that the existence of an associative coding function suffices for standard sense of direction:

Theorem 2 *[6] Any associative coding function has a consistent decoding.*

In the case of everywhere associative coding functions, the resulting decoding function has the additional property of being everywhere consistent:

Theorem 3 *Any everywhere associative coding function has a everywhere consistent decoding.*

Proof Let f be an everywhere consistent coding function in (G, λ, β).
Consider the following decoding function: $\forall l \in \mathcal{L}, \forall n \in \mathcal{N} \subseteq \mathcal{L}^+.\ h(l, n) = f(l \cdot n)$.
We now show that it is everywhere consistent.
By definition of h, $\forall l \in \mathcal{L}$ and $\forall \alpha \in \mathcal{L}^+$, $h(l, f(\alpha)) = f(l \cdot f(\alpha))$.
By definition of everywhere associativity, we have that: $\forall l \in \mathcal{L}$ and $\forall \alpha \in \mathcal{L}^+$, $f(l \cdot f(\alpha)) = f(l \cdot \alpha)$. It follows that: $\forall l \in \mathcal{L}$ and $\forall \alpha \in \mathcal{L}^+$, $h(l, f(\alpha)) = f(l \cdot \alpha)$, which is the definition of everywhere consistency. □

Summarizing, (everywhere) associativity of the coding function implies sense of direction. We will now show that something similar to the converse also holds: for any system with sense of direction there is a coding function associative everywhere. Before stating the theorem, some terminology and definitions are in order.

Definition 8 Selection Function
Given a set S*, a function* $s : 2^S \to S$ *is a* selection function *on* S *iff* $\forall A \subseteq S,\ s(A) \in A$.

Definition 9 Selection Inverse
Given a coding function $f : \mathcal{L} \to \mathcal{N}$ *and a selection function* s *on* $\mathcal{N}$*, the* selection inverse *of* f *is* $f_s^{-1} = s \circ f^{-1}$*, s.t.* $f^{-1}(n) = \{\Lambda_x(\pi) : \forall x, \pi \in P[x], f(\Lambda_x(\pi)) = n\}$.

From the above definitions, the following property holds.

Property 1 *For each f and each s:*
(i) $f \circ f_s^{-1} \circ f = f$
(ii) if $f_s^{-1} \circ f(\alpha) = f_s^{-1} \circ f(\beta)$, then $f(\alpha) = f(\beta)$.

We can now state and prove the theorem. The proof is constructive.

Theorem 4 *If (G, λ, β) has a sense of direction, there exists a $\mathcal{SD}$ (f', h') with f' everywhere associative and h' everywhere consistent for f'.*

Proof Let (f, h) be any sense of direction in (G, λ, β). By Theorem 1, we can assume that h is everywhere consistent for f. Let $f' = f_s^{-1} \circ f$. The coding function f' is consistent because, by Property 1 (ii), $\forall x, y, z \in V$ and $\forall \pi_1 \in P[x, y], \forall \pi_2 \in P[x, z]$,

$$f_s^{-1} \circ f(\Lambda_x(\pi_1)) = f_s^{-1} \circ f(\Lambda_x(\pi_2)) \text{ iff } f(\Lambda_x(\pi_1)) = f(\Lambda_x(\pi_1))$$

Furthermore, by consistency of f, $f(\Lambda_x(\pi_1)) = f(\Lambda_x(\pi_1))$ iff $y = z$.
Now we prove that $f_s^{-1} \circ f$ is everywhere associative. Since f is everywhere consistent, for each $a \in \mathcal{L}$ and each $w \in \mathcal{L}^+$

$$f'(a \cdot f'(w)) = f_s^{-1}(f(a \cdot f_s^{-1}(f(w)))) = f_s^{-1}(h(a, f(f_s^{-1}(f(w))))$$

By Property 1 (i) and by everywhere consistency of h, we have

$$f_s^{-1}(h(a, f(f_s^{-1}(f(w)))) = f_s^{-1}(h(a, f(w))) = f_s^{-1}(f(a \cdot w)) = f'(a \cdot w)$$

Let $h'(a, f'(w)) = f'(a \cdot f'(w))$, by everywhere associativity of f', $f'(a \cdot f'(w)) = f'(a \cdot w)$, thus h' is an everywhere consistent decoding of f'. □

4 Symmetric Sense of Direction

We now focus on systems with edge symmetry that is, where there is a relationship between the two labels given by the nodes incident on the same edge. For such systems we consider some restrictions on the naming functions leading to a stronger form of Sense of Direction.

4.1 Edge and Name Symmetry

Definition 10 $\mathcal{ES}$ - Edge Symmetry
A labeling λ has Edge Symmetry *($\mathcal{ES}$) iff there exists a bijection $\psi : \mathcal{L} \to \mathcal{L}$, such that, $\forall \langle x, y \rangle \in E$, $\lambda_x(\langle x, y \rangle) = \psi(\lambda_y(\langle y, x \rangle))$.*

In this case, we will also say that the system has edge symmetry, and call ψ the edge symmetry function.

Let $\Psi : \mathcal{L}^+ \to \mathcal{L}^+$ be the extention of the edge symmetry function from edges to path, that is: $\Psi(\alpha) = [\psi(\alpha_1), \ldots, \psi(\alpha_m)]$, where $\alpha = [\alpha_1, \ldots, \alpha_m]$ is a sequence of labels corresponding to a path in G. Given a path π, we denote by $\overline{\pi}$ the reversed path. By definition of Ψ, we have that:

Property 2 $\forall \pi \in P[x,y]$, $\Lambda_x(\pi) = \Psi(\Lambda_y(\overline{\pi}))$

Definition 11 $\mathcal{NS}$ - Name Symmetry
A naming function β *has* Name Symmetry *($\mathcal{NS}$) iff there exist a bijection* $\mu : \mathcal{N} \to \mathcal{N}$*, such that,* $\forall x, y \in V$, $\beta_x(y) = \mu(\beta_y(x))$*.*

In this case, we will also say that the system has name symmetry, and call μ the name symmetry function.

Note that name symmetry does not necessarily imply edge symmetry. Consider in fact the a labeled graph (G, λ) shown in Figure 2 (B); λ does not have edge symmetry. However, using as β the "chordal" naming function, (G, λ, β) has the "chordal" sense of direction [6].

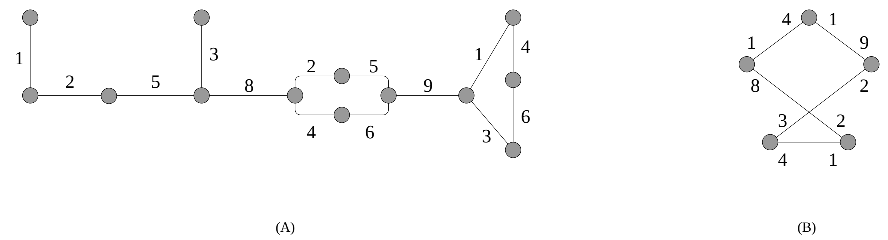

Figure 2: Systems (A) with $\mathcal{ES}$, without $\mathcal{NS}$, (B) with $\mathcal{NS}$, without $\mathcal{ES}$.

Symmetric sense of direction is defined in systems with edge and name symmetry.

Definition 12 $\mathcal{SSD}$ - Symmetric Sense of Direction
A system (G, λ, β) *has* Symmetric Sense of Direction *iff it has sense of direction and both edge symmetry and name symmetry.*

Notice that almost all the systems studied in the literature in the context of sense of direction are actually symmetric ones.

4.2 Sense of Direction in Presence of Edge Symmetry

In this section we consider only systems with edge symmetry. For these systems we study under which conditions there is name symmetry; i.e., when a sense of direction is symmetric. In the following, all systems (G, λ, β) are edge-symmetric.

The following theorem provides a characterization of name symmetry in terms solely of the coding function. Let $\circ$ denote the composition of functions.

Theorem 5 *Let* f *be a* $\mathcal{WSD}$ *in* (G, λ, β)*. The system has name symmetry iff* $\forall \pi_1 \in P[s,t]$, $\forall \pi_2 \in P[w,z]$, $f(\Lambda_s(\pi_1)) = f(\Lambda_w(\pi_2)) \Rightarrow f(\Psi(\Lambda_s(\pi_1))) = f(\Psi(\Lambda_w(\pi_2)))$.

Proof ($\Rightarrow$) By contradiction, suppose that there exists a name symmetry function μ and that there exist $\pi_1 \in P[s,t], \pi_2 \in P[w,z]$ s.t. $f(\Lambda_s(\pi_1)) = f(\Lambda_w(\pi_2))$ while $f(\Psi(\Lambda_s(\pi_1))) \neq f(\Psi(\Lambda_w(\pi_2)))$.
Since $\overline{\pi_1} \in P[t,s]$ and $\overline{\pi_2} \in P[z,w]$, by Property 2, $\Lambda_t(\overline{\pi_1}) = \Psi(\Lambda_s(\pi_1))$ and $\Lambda_z(\overline{\pi_2}) = \Psi(\Lambda_w(\pi_2))$. By name symmetry, it must be $\mu(f(\Lambda_s(\pi_1))) = f(\Lambda_t(\overline{\pi_1})) = f(\Psi(\Lambda_s(\pi_1)))$ and $\mu(f(\Lambda_w(\pi_2))) = f(\Lambda_z(\overline{\pi_2})) = f(\Psi(\Lambda_w(\pi_2)))$.
But $f(\Lambda_s(\pi_1)) = f(\Lambda_w(\pi_2))$ implies $\mu(f(\Lambda_s(\pi_1))) = \mu(f(\Lambda_w(\pi_2)))$, which in turn implies $f(\Psi(\Lambda_s(\pi_1))) = f(\Psi(\Lambda_w(\pi_2)))$, a contradiction.
($\Leftarrow$) We first prove that $f \circ \Psi \circ f_s^{-1} \circ f = f \circ \Psi$. By Property 1, $f \circ f_s^{-1} \circ f = f$. By hypothesis, $f \circ \Psi \circ f_s^{-1} \circ f = f \circ \Psi$. Let $\mu = f \circ \Psi \circ f_s^{-1}$; then $\mu(f(\Lambda_x(\pi))) = f(\Psi(f_s^{-1}(f(\Lambda_x(\pi))))) = f(\Psi(\Lambda_x(\pi))) = f(\Lambda_y(\overline{\pi}))$. □

Clearly, any symmetric sense of direction is also standard. We will now prove that the hierarchical relationship between standard and symmetric sense of direction is strict even when restricted to edge-symmetric systems: there are edge-symmetric systems with sense of direction but without name symmetry. There are actually graphs with edge-symmetric labelings in which *every* naming function has *no* name symmetry.

Theorem 6 *Edge symmetry is not sufficient for* $\mathcal{SSD}$

Proof Consider the graph (G,λ) of Figure 2 (A) where the labeling is a coloring, and thus has edge symmetry. It is easy to see that there exists a β and a $\mathcal{SD}$ (f,h) in (G,λ,β). We show that, for any sense of direction, there is no name symmetry.
By consistency of f, we have that $f(2 \cdot 5) = f(4 \cdot 6)$, thus $h(1, f(2 \cdot 5)) = h(1, f(4 \cdot 6))$, which implies that $f(1 \cdot 2 \cdot 5) = f(1 \cdot 4 \cdot 6)$. On the other hand, by consistency of f, we have that $f(5 \cdot 2 \cdot 1) \neq f(3)$, and $f(6 \cdot 4 \cdot 1) = f(3)$, which implies that $f(5 \cdot 2 \cdot 1) \neq (6 \cdot 4 \cdot 1)$. By Theorem 5 it follows that there is no name symmetry in the system. □

We now study a property, *backward decoding*, which has a strong relationship with symmetric sense of direction. In the following, all systems are edge-symmetric.

Definition 13 Backward Decoding Function
Given a coding function f, *a* backward decoding function *for* f *is any function* $h_b : \mathcal{N} \times \mathcal{L} \rightarrow \mathcal{N}$ *s.t.* $\forall \pi \in P[x,y]$, $\langle y,z \rangle \in E(y)$,

$$h_b(f(\Lambda_x(\pi)), \lambda_y(\langle y,z \rangle)) = f(\Lambda_x(\pi) \cdot \lambda_y(\langle y,z \rangle))$$

The existence of a backward consistent decoding function is crucial in "wireless" systems where there is not even local orientation [8].

We now show that, in systems with sense of direction, the existence of a backward consistent decoding function is sufficient for the existence of a $\mathcal{SSD}$.

Theorem 7 *Let* (f,h) *be a sense of direction in* (G,λ,β) *with edge symmetry. If there exists a backward decoding function* h_b *for* f, *then there exists a* $\mathcal{SSD}$ $(\overline{f},\overline{h})$ *and a backward decoding function* $\overline{h_b}$ *for* $\overline{f}$.

Proof Let $\overline{f} : \mathcal{L}^+ \to 2^{\mathcal{L}^+}$ be defined as follows:

$$\overline{f}(w) = f^{-1} \circ f(w) \cap \Psi \circ f^{-1} \circ f \circ \Psi(w).$$

Claim 1 *$\alpha \in \overline{f}(w)$ iff $f(\alpha) = f(w)$ and $f \circ \Psi(\alpha) = f \circ \Psi(w)$.*

Proof (of **Claim 1**)
$f^{-1} \circ f(w) = \{\alpha : f(\alpha) = f(w)\}$ and $\Psi \circ f^{-1} \circ f \circ \Psi(w) = \Psi(\{\alpha' : f(\alpha') = f \circ \Psi(w)\}) = \{\Psi(\alpha') : f(\alpha') = f \circ \Psi(w)\} = \{\Psi \circ \Psi(\alpha) : \Psi(\alpha) = \alpha' \wedge f(\alpha') = f \circ \Psi(w)\} = \{\alpha : f \circ \Psi(\alpha) = f \circ \Psi(w)\}$. Thus $\overline{f}(w) = \{\alpha : f(\alpha) = f(w)\} \cap \{\alpha' : f \circ \Psi(\alpha') = f \circ \Psi(w)\} = \{\alpha : f(\alpha) = f(w) \wedge f \circ \Psi(\alpha) = f \circ \Psi(w)\}$. $\square$

Claim 2 *For each selection function s:*

$$f \circ s \circ \overline{f} = f \quad \textit{and} \quad f \circ \Psi \circ s \circ \overline{f} = f \circ \Psi.$$

Proof (of **Claim 2**)
$s \circ \overline{f}(w) \in \overline{f}(w)$ by definition of selection function. By Property 1, $f \circ s \circ \overline{f}(w) = f(w)$ and $f \circ \Psi \circ s \circ \overline{f}(w) = f \circ \Psi(w)$. $\square$

We now show that $\overline{f}$ is consistent. $\forall x, y, z, \pi_1 \in P[x, y], \pi_2 \in P[x, z]$ $\overline{f}(\Lambda_x(\pi_1)) = \overline{f}(\Lambda_x(\pi_2))$ implies $f(\Lambda_x(\pi_1)) = f(\Lambda_x(\pi_2))$ by Claim 1. By consistency of f, it follows that $y = z$. Thus we have proved that $\overline{f}(\Lambda_x(\pi_1)) = \overline{f}(\Lambda_x(\pi_2))$ implies $y = z$. If $y = z$, then $\overline{\pi_1}, \overline{\pi_2} \in P[y, x]$. Thus, by consistency of f, $f(\Lambda_y(\overline{\pi_1})) = f(\Lambda_y(\overline{\pi_2}))$. By property 2, $\Lambda_y(\overline{\pi_1}) = \Psi(\Lambda_x(\pi_1))$ and $\Lambda_y(\overline{\pi_2}) = \Psi(\Lambda_x(\pi_2))$. Then $\overline{f}(\Lambda_x(\pi)) = f^{-1} \circ f(\Lambda_x(\pi_1)) \cap \Psi \circ f^{-1} \circ f \circ \Psi(\Lambda_x(\pi_1)) = f^{-1} \circ f(\Lambda_x(\pi_2) \cap \Psi \circ f^{-1} \circ f \circ \Psi(\Lambda_x(\pi_2)) = \overline{f}(\Lambda_x(\pi_2))$, showing that $\overline{f}$ is consistent.
Let $\overline{h} : \mathcal{L} \times 2^{\mathcal{L}^+} \to 2^{\mathcal{L}^+}$ be defined as follows:

$$\overline{h}(a, \overline{f}(w)) = f^{-1}(h(a, f \circ s(\overline{f}(w))) \cap \Psi \circ f^{-1}(h_b(f \circ \Psi \circ s(\overline{f}(w), \psi(a)))))$$

where s is a selection function.
Now we prove that $\overline{h}$ is a consistent decoding function for $\overline{f}$. That is, $\forall x, y, z$, $\forall \langle x, y\rangle\pi \in P[x, z]$, $\overline{h}(\lambda_x(\langle x, y\rangle), \overline{f}(\Lambda_y(\pi))) = \overline{f}(\Lambda_x(\langle x, y\rangle\pi))$.
By definition of $\overline{h}$, $\overline{h}(\lambda_x(\langle x, y\rangle), \overline{f}(\Lambda_y(\pi))) = f^{-1}(h(\lambda_x(\langle x, y\rangle), f \circ s(\overline{f}(\Lambda_y(\pi))) \cap \Psi \circ f^{-1}(h_b(f \circ \Psi \circ s(\overline{f}(\Lambda_y(\pi)), \psi(\lambda_x(\langle x, y\rangle)))))))$.
By Claim 2, $f^{-1}(h(\lambda_x(\langle x, y\rangle), f \circ s(\overline{f}(\Lambda_y(\pi))) \cap \Psi \circ f^{-1}(h_b(f \circ \Psi \circ s(\overline{f}(\Lambda_y(\pi)), \psi(\lambda_x(\langle x, y\rangle)))))) = f^{-1}(h(\lambda_x(\langle x, y\rangle)\ f(\Lambda_y(\pi))) \cap \Psi \circ f^{-1}(h_b(f \circ \Psi(\Lambda_y(\pi)), \psi(\lambda_x(\langle x, y\rangle)))))$.
By consistency of h and h_b, $f^{-1}(h(\lambda_x(\langle x, y\rangle), f(\Lambda_y(\pi))) \cap \Psi \circ f^{-1}(h_b(f \circ \Psi(\Lambda_y(\pi)), \psi(\lambda_x(\langle x, y\rangle))))) = f^{-1}(f(\lambda_x(\langle x, y\rangle) \cdot \Lambda_y(\pi))) \cap \Psi \circ f^{-1}(\Psi(\Lambda_y(\pi)) \cdot \psi(\lambda_x(\langle x, y\rangle))))$.
Finally, $f^{-1}(f(\lambda_x(\langle x, y\rangle) \cdot \Lambda_y(\pi))) \cap \Psi \circ f^{-1}(\Psi(\Lambda_y(\pi)) \cdot \psi(\lambda_x(\langle x, y\rangle))))$
$= f^{-1}(f(\Lambda_x(\langle x, y\rangle\pi))) \cap \Psi \circ f^{-1}(\Psi(\Lambda_x(\langle x, y\rangle\pi))) = \overline{f}(\Lambda_x(\langle x, y\rangle\pi))$.
Thus $\overline{h}$ is consistent.
In a similar way, it can be shown that the function $\overline{h_b}(\overline{f}(w), a) = f^{-1}(h_b(\overline{f}(w), a)) \cap \Psi \circ f^{-1}(h(\psi(a), f \circ \Psi \circ s(\overline{f}(w))))$ is a backward consistent decoding function for $\overline{f}$.

We now prove that $\mu = \Psi : 2^{\mathcal{L}} \rightarrow 2^{\mathcal{L}}$ is the symmetry function. That is, $\forall x, y, \pi \in P[x,y]$, $\Psi(\overline{f}(\Lambda_x(\pi)) = \overline{f}(\Lambda_y(\overline{\pi}))$. By Property 2, $\overline{f}(\Lambda_y(\overline{\pi})) = \overline{f}(\Psi(\Lambda_x(\pi)))$. By definition of $\overline{f}$,
$\overline{f}(\Psi(\Lambda_x(\pi))) = f^{-1} \circ f(\Psi(\Lambda_x(\pi))) \cap \Psi \circ f^{-1} \circ c \circ \Psi(\Psi\Lambda_x(\pi))$.
By idempotency of Ψ, it follows $f^{-1} \circ f(\Psi(\Lambda_x(\pi))) \cap \Psi \circ f^{-1} \circ f \circ \Psi(\Psi\Lambda_x(\pi))$ $= \Psi \circ \Psi \circ f^{-1} \circ f(\Psi(\Lambda_x(\pi))) \cap \Psi \circ f^{-1} \circ f \circ \Lambda_x(\pi)$.
By definition of $\overline{f}$, $\Psi(\Psi \circ f^{-1} \circ f(\Psi(\Lambda_x(\pi))) \cap f^{-1} \circ f \circ \Lambda_x(\pi)) =$
$\Psi \circ \overline{f}(\Lambda_x(\pi))$ which concludes the proof. consistent decoding function for $\overline{f}$.
We now prove that $\mu = \Psi : 2^{\mathcal{L}} \rightarrow 2^{\mathcal{L}}$ is the symmetry function. That is, $\forall x, y, \pi \in P[x,y]$, $\Psi(\overline{f}(\Lambda_x(\pi)) = \overline{f}(\Lambda_y(\overline{\pi}))$. By Property 2, $\overline{f}(\Lambda_y(\overline{\pi})) = \overline{f}(\Psi(\Lambda_x(\pi)))$. By definition of $\overline{f}$,
$\overline{f}(\Psi(\Lambda_x(\pi))) = f^{-1} \circ f(\Psi(\Lambda_x(\pi))) \cap \Psi \circ f^{-1} \circ c \circ \Psi(\Psi\Lambda_x(\pi))$.
By idempotency of Ψ, it follows
$f^{-1} \circ f(\Psi(\Lambda_x(\pi))) \cap \Psi \circ f^{-1} \circ f \circ \Psi(\Psi\Lambda_x(\pi)) = \Psi \circ \Psi \circ f^{-1} \circ f(\Psi(\Lambda_x(\pi))) \cap$ $\Psi \circ f^{-1} \circ f \circ \Lambda_x(\pi)$.
By definition of $\overline{f}$, $\Psi(\Psi \circ f^{-1} \circ f(\Psi(\Lambda_x(\pi))) \cap f^{-1} \circ f \circ \Lambda_x(\pi)) =$
$\Psi \circ \overline{f}(\Lambda_x(\pi))$ which concludes the proof. □

In other words, in any system (G, λ, β) where there exists a sense of direction with a backward decoding , there exists also a symmetric sense of direction.

We will now show that symmetric sense of direction implies the existence of backward decoding. We first prove the following useful property of name symmetry functions:

Property 3 *Let f be a coding function with name symmetry and let μ be the name symmetry function. For any $\alpha \in \mathcal{L}^+$ corresponding to a path:* $\mu(\mu(f(\alpha)) = f(\alpha)$

Proof By definition of name symmetry function, we have: $\mu(\mu(f(\alpha))) = \mu(f(\Psi(\alpha))) = f(\Psi(\Psi(\alpha))) = f(\alpha)$. □

Theorem 8 *Backward decoding is necessary for $\mathcal{SSD}$.*

Proof Let (f, h) be a $\mathcal{SSD}$ and let ψ and μ be respectively the edge and name symmetry functions. Let $x, y \in V$, $\pi \in P[x,y]$, and $\omega = \Lambda_x(\pi)$. By definition of name symmetry function, $f(\Psi(\omega)) = \mu(f(\omega))$, it follows that $\mu(f(\Psi(\omega))) = \mu(\mu(f(\omega)))$. By Property 3, we have that

$$\mu(f(\Psi(\omega))) = f(\omega) \qquad (1)$$

Consider now the following backward decoding function: $\forall \pi \in P[x,y]$, $\omega = \Lambda_x(\pi)$ and for $\langle y, z\rangle = a \in E(y)$, $h_b(f(\omega), a) = \mu(h(\psi(a), \mu(f(\omega)))$. By definition of name symmetry function $\mu(f(\omega)) = f(\Psi(\omega))$, thus,
$\mu(h(\psi(a), \mu(f(\omega)) = \mu(h(\psi(a), f(\Psi(\omega))) = \mu(f(\Psi(w \cdot a))$.
By (1), $\mu(f(\Psi(w \cdot a)) = f(\omega \cdot a)$. It follows that $h_b(f(\omega), a) = f(\omega \cdot a)$, thus, h_b is consistent. □

5 Conclusions

In this paper we have studied the (standard) Sense of Direction ($\mathcal{SD}$), as well as two other levels of global consistency constraints: a weak form, called Weak Sense of Direction ($\mathcal{WSD}$), and a stronger form, called Symmetric Sense of Direction ($\mathcal{SSD}$). In particular, we have studied these forms of sense of direction with respect to a set of consistency properties: *associativity*, *backward decoding*, *everywhere consistency*, and *name symmetry*.

Sense of direction has been studied extensively and it has been shown that its properties can be heavily exploited for reducing the communication complexity of many important problems. An interesting problem is now to study what kind of reductions can be obtained when weaker (or stronger) forms of sense of direction are available. Some questions arise naturally:

What are the limits of sense of direction which could be overcome by a stronger labeling (e.g., $\mathcal{SSD}$)? Which problems become "easier" when a $\mathcal{SSD}$ is available?

Which useful properties are preserved even with weaker (and, thus, less expensive to construct) forms of consistency constraints (e.g., $\mathcal{WSD}$) ? Some results in this direction have been obtained in [9].

Are there other labelings with different properties which, when available, can be similarly exploited for obtaining the same cost reduction ?

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