Minimal Sense of Direction in Regular Networks*

Paola Flocchini

Département d' Informatique et de Recherche Opérationnelle

Université de Montréal

Montréal, H3C 3J7

Canada

Abstract

A network is said to have *Sense of Direction* when the port labeling satisfies a particular set of global consistency constraints. In this paper we study the link between the topology of a system and the number of labels that are necessary to have a Sense of Direction in that system. We consider systems whose topology is a *regular graph* and we study the relationship between structural properties of d-regular graphs and existence of a Sense of Direction which uses exactly d labels (*minimal* $\mathcal{SD}$). In particular, we identify a property (*Cycle Symmetricity*) which we show is a necessary condition for minimal $\mathcal{SD}$. Among regular graphs, we then focus on *Cayley Graphs* and we prove that they *always* have a minimal Sense of Direction.

Keywords. distributed computing, sense of direction, graph labelings, Cayley graphs.

1 Introduction

A *distributed system* is a collection of processing entities (e.g., processors) connected by a communication network, where each entity has a local non-shared memory and can communicate by sending messages to and receiving messages from its neighbors. Every entity has a distinct label (e.g., port number) associated with each of its incident links. Thus, the entire system can be viewed as a graph where each node corresponds to a system entity, and each edge corresponds to a direct communication link between two entities; furthermore, every edge has two labels, one for each of its incident nodes. A classical example is a ring network where each edge is labeled "right" at an incident node and "left" at the other.

The system is said to have *Sense of Direction* if the edge-labeling satisfies a particular set of global consistency constraints [9]. It is well known that the presence of Sense of Direction can have a dramatic effect on the performance of communication protocols for a large class

*Research supported in part by a CNR-NATO Advanced Research Fellowship and by an international CNR Scholarship.

of distributed problems (e.g., see [2, 8, 13, 14, 15, 19, 20, 21, 23, 22]). Because of this impact on complexity, the characterization and analysis of what constitutes Sense of Direction is an important and practical task.

Most of the existing research has focused on the study of specific labelings in specific topologies: the "left-right" labeling of rings (e.g., [3, 12]), *chordal* labelings in complete networks and chordal rings (e.g., [2, 24]), *dimensional* labelings in hypercubes (e.g., [8, 26]), *compass* labelings in meshes and tori (e.g., [4, 25, 28]), *contracted* labeling in most interconnection networks, etc.

Sense of Direction has also been studied at a more general level, focusing, for example, on the "semantic" capabilities of $\mathcal{SD}$ [10], on the analysis of the complexity of determining whether a labeling is a $\mathcal{SD}$ [6] and on the study of the interplay existing between the structure of the system and properties which a labeling must satisfy to be a $\mathcal{SD}$ [11].

In this paper we study the link between the topology of a system and the number of labels that are necessary to have a Sense of Direction in that system. In a network with maximum degree d, the number of labels necessary for a $\mathcal{SD}$ is at least d, since any node must be able to distinguish among its incident edges. A $\mathcal{SD}$ which uses exactly d labels is said to be *minimal*. Not every network has a minimal $\mathcal{SD}$; thus, an interesting open problem is to determine the topological conditions for its existence.

In this paper, we consider systems whose topology is a *regular graph* and we study the relationship between structural properties of regular graphs and existence of a minimal Sense of Direction. In particular, we identify a property (*Cycle Symmetricity*) which we show is a necessary condition for minimal $\mathcal{SD}$. Unfortunately this condition is, in general, not sufficient; thus, the complete characterization of regular graphs with minimal $\mathcal{SD}$ is still open.

Among regular graphs, we then focus on *Cayley Graphs* and we prove that they *always* have a minimal Sense of Direction.

In the next Section we give some basic definitions, in Section 3 we study the conditions under which it is possible to have a minimal $\mathcal{SD}$, in Section 4 we consider Cayley graphs, and we show that their natural labeling is always a minimal Sense of Direction.

2 Labelings and Sense of Direction

In a distributed system with *Local Orientation*, each system entity can distinguish among its neighbours; in practice, an entity enforces this ability by assigning a distinct label (called *port number*) to each of its communication links. Each node of the system also associates a *local name* to each one of the other nodes.

Let $G = (V, E)$ be a graph where nodes correspond to entities and edges correspond to direct bidirectional communication links between entities. Let $E(x)$ denote the set of edges incident to node x.

A distributed system can, thus, be described as a triple (G, λ, β), where: G is the graph describing the communication topology of the system, $\lambda = \{\lambda_x : x \in V\}$ is the set of labeling functions, where $\lambda_x(\langle x, z \rangle)$ denotes the label, in a finite set of labels $\mathcal{L}$, associated by x to

$\langle x, z \rangle \in E(x)$; $\beta = \{\beta_x : x \in V\}$ is the set of naming functions, where $\beta_x(y)$ denotes the name that node x associates to node y from a finite set of names $\mathcal{N}$.

A *path* in G is a sequence of edges $[\langle x_0, x_1 \rangle, \langle x_1, x_2 \rangle, \ldots, \langle x_{m-1}, x_m \rangle]$, $\langle x_i, x_{i+1} \rangle \in E(x_i)$, in which the endpoint of one edge is the starting point of the next edge. Let $P[x]$ denote the set of all paths with $x \in V$ as a starting point, and let $P[x, y]$ denote the set of paths starting from node $x \in V$ and ending in node $y \in V$. Let Λ be the extension of the labeling function λ from edges to paths.

A labeling λ is a *Local Orientation* when each node can distinguish among its incident edges; that is when, $\forall x \in V$, $\forall e_1, e_2 \in E(x)$, $\lambda_x(e_1) = \lambda_x(e_2)$ iff $e_1 = e_2$.

A *consistent coding function* f of a graph (G, λ, β) is a a function that maps the sequences of labels associated to any paths from x to y, to the local name $\beta_x(y)$ used by x to refer to y. More formally,

Definition 1 Consistent Coding Function
A coding function f is consistent *in (G, λ, β) iff $\forall x, y \in V$, $\pi \in P[x, y]$, $f(\Lambda_x(\pi)) = \beta_x(y)$.*

A *consistent decoding function* h for f is a function that associates a name to a given name and a label and allows a node to translate the local views of its neighbours.

Definition 2 Consistent Decoding Function
Given a consistent coding function f, a decoding function h for f is consistent iff $\forall \langle x, y \rangle \in E(x), \pi \in P[y, z]$ $h(\lambda_x(\langle x, y \rangle), f(\Lambda_y(\pi))) = \beta_x(z)$.

Definition 3 *[9] $\mathcal{SD}$* - Sense of Direction
Given (G, λ, β), λ is a Sense of Direction *($\mathcal{SD}$) iff the following conditions hold:*
1) λ is a Local Orientation,
2) there exists a consistent coding function f,
3) there exists a consistent decoding function h for f.

Given a labeled graph (G, λ), we say that λ is *globally consistent* iff there exists a β such that λ is a $\mathcal{SD}$ in (G, λ, β).

It has been observed before that an important problem is how to build a $\mathcal{SD}$ which uses the minimum number of labels. Obviously, any $\mathcal{SD}$ must use at least d labels, where d is the maximum degree of the graph; it is possible to build a $\mathcal{SD}$ in any graph using n labels (for example Chordal, Neighbouring [9]), where n is the number of nodes. Thus, the number of labels used lies between d and n; $d \leq |\mathcal{L}| \leq n$.
Let $\mathcal{L}$ be the set of labels used by a labeling λ.

Definition 4 *The* size *of a labeling λ is the number of different labels in $\mathcal{L}$.*

Definition 5 Minimal $\mathcal{SD}$
A Sense of Direction λ in (G, λ, β) is minimal *iff its size is equal to the degree of G.*

3 Cycle Symmetricity

In this section we present a necessary condition for having minimal $\mathcal{SD}$ in regular graphs.

Determing the existence of a minimal $\mathcal{SD}$ in a regular graph is sometimes an easy task; consider, for example, the regular graph of Figure 1 *a*). In this case it is clearly possible to construct a $\mathcal{SD}$ using only 4 labels, e.g. the classical compass labeling [9]. On the other hand, there are graphs for which it is not immediate that one can find a minimal labeling without violating the properties of $\mathcal{SD}$, for example the one in Figure 1 *b*). Actually, as we will see later, for such a graph there is no minimal $\mathcal{SD}$.

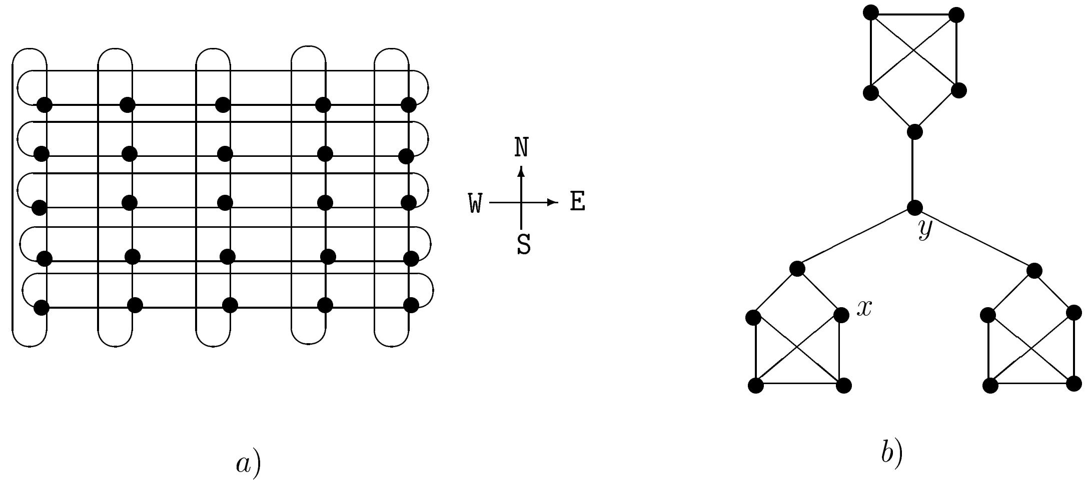

Figure 1: Regular graphs for which *a*) there exists a minimal $\mathcal{SD}$, *b*) there exists no minimal $\mathcal{SD}$.

We first define two properties of symmetricity in regular graphs, *vertex symmetricity* and *cycle symmetricity*, which characterize interesting classes of graphs.

Let $G = (V, E)$ be a regular graph. A *vertex automorphism* α of G is a permutation of the vertex set that preserves the adjacency, i.e., $\langle \alpha(x), \alpha(y) \rangle \in E$ if $\langle x, y \rangle \in E$.

Definition 6 Vertex Symmetricity.
G is vertex symmetric *iff* $\forall x, y \in V$ *there exists an automorphism* α *of the graph, such that* $y = \alpha(x)$.

In other words, when a graph is vertex symmetric, the graph looks the same viewed from any node. Vertex symmetricity is an important property, and it is often exploited in the design of interconnection networks (e.g. [1, 7, 18]).

We now consider another property of symmetricity in regular graphs, called *cycle symmetricity*.

For any node $x \in V$, denote with $C_x[i]$ the number of cycles of length i to which x belongs.

Definition 7 Cycle Symmetricity.
A graph is cycle symmetric *iff*

$$\forall i \in \mathbf{N},\ \forall x, y \in V,\ \ C_x[i] = C_y[i]$$

In other words, a graph is cycle symmetric when all the nodes belong to the same number of cycles of the same length.

The next Theorem shows that, if a graph is vertex symmetric, then it is also cycle symmetric.

Theorem 1 *Vertex symmetricity implies cycle symmetricity.*

Proof Let G be a vertex symmetric graph. Let $x_0 \in V$ and $[x_0, \ldots, x_k, x_0] \in P[x_0, x_0]$. Consider any other node $y_0 \neq x_0$. By definition of vertex symmetricity, there exists an automorphism of the graph α such that $y_0 = \alpha(x_0)$, by definition of automorphism, there must exist a neighbour y_1 of y_0 such that $y_1 = \alpha(x_1)$. In general the entire cycle $[x_0, \ldots, x_k, x_0]$ is transformed by α in a cycle $[y_0, \ldots, y_k, y_0] = [\alpha(x_0), \ldots, \alpha(x_k), \alpha(x_0)]$. Thus, if a node x_0 belongs to a cycle of length k, any other node y_0 belongs to a cycle of length k as well; thus, the graph is cycle symmetric. □

Theorem 2 Cycle symmetricity *is necessary for minimal $\mathcal{SD}$ in regular graphs.*

Proof By contradiction. Consider a graph G that is not cycle symmetric and suppose that G has a minimal $\mathcal{SD}$ λ. Since G is not cycle symmetric,

$$\exists x, y \in V, k \in \mathbf{N} : C_x[k] \neq C_y[k] \tag{1}$$

Without loss of generality, let $C_x[k] > C_y[k]$. Let $\mathcal{C}_x^k$ be the set of cycles of length k to which x belongs and $\mathcal{C}_y^k$ the set of cycles of length k to which y belongs.
We could have, either $i)$ $C_x[k] > C_y[k] = 0$, or $ii)$ $C_x[k] > C_y[k] > 0$.
Case $i)$. Let $c \in \mathcal{C}_x^k$, $c = [\langle x, x_1\rangle, \langle x_1, x_2\rangle, \ldots, \langle x_m, x\rangle]$. Let $\lambda_x(\langle x, x_1\rangle) = a$, and $\Lambda_x(\langle x, x_m\rangle, \langle x_m, x_{m-1}\rangle, \ldots, \langle x_2, x_1\rangle) = \alpha$. By definition of consistent coding function, we have that:

$$f(a) = f(\alpha). \tag{2}$$

Since the graph is d-regular and λ is minimal, there must exist an edge $\langle y, y'\rangle$ such that $\lambda_y(\langle y, y'\rangle) = a$, and a path of length $k-1$ starting from y, $\pi \in P[y]$, such that $\Lambda_y(\pi) = \alpha$. Since $C_y[k] = 0$, $[\langle y', y\rangle \cdot \pi]$ (where $\cdot$ is the concatenation of two paths) cannot be a cycle, thus, it must be:

$$f(a) \neq f(\alpha)$$

which contradicts (2).
Case $ii)$: $C_x[k] > C_y[k] > 0$. In this case, there are clearly $C_x[k] - C_y[k]$ cycles passing through x, to which the same argument of case $i)$ can be applied. □

Figure 1 shows two regular graphs. Graph $b)$ is not cyclic symmetric; in fact, for example, node x belongs to a cycle of length 5 while node y does not belong to a cycle of such length. It is easy to see that the torus (case $a)$) is cycle symmetric.

We will now show that cycle symmetricity is not sufficient for minimal $\mathcal{SD}$ in regular graphs; before doing that we introduce some definitions and a Lemma.

A labeling has *Edge Consistency* when there exists a relation between the labels at the two sides of each edge. More precisely,

Definition 8 Edge Consistency
A labeling has edge consistency *iff there exists a consistency function* $\psi : \mathcal{L} \to \mathcal{L}$*, such that,* $\forall \langle x, y\rangle \in E$, $\lambda_x(\langle x, y\rangle) = \psi(\lambda_y(\langle y, x\rangle))$.

Notice that sometimes this propetry is known as *Edge Symmetry* (e.g., [9, 11])

Let $\Psi : \mathcal{L}^* \to \mathcal{L}^*$ be the extension of the edge consistency function from edges to path.

Definition 9 Name Consistency
A naming function β *has* name consistency *if there exists a function* $\mu : \mathcal{N} \to \mathcal{N}$*, such that,* $\forall x, y \in V$, $\beta_x(y) = \mu(\beta_y(x))$.

Consider now Petersen's graph (see Figure 2); in this case we have that:

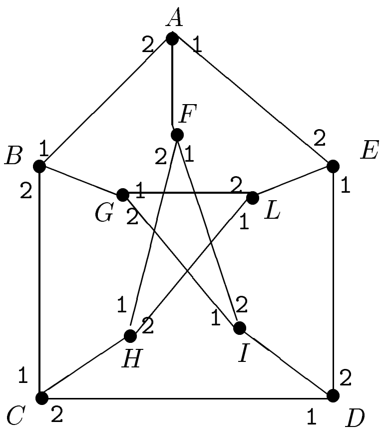

Figure 2: Petersen's graph. The arcs which have no labels in the figure, are labeled with "3" on both sides.

Lemma 1 *Any minimal* $\mathcal{SD}$ *in Petersen's graph, must have edge consistency.*

Proof By contradiction. Let λ be a minimal $\mathcal{SD}$ in Petersen's graph without edge consistency. Since there isn't edge consistency, and λ uses only three labels there must exist two edges $\langle x, y\rangle$, $\langle y, z\rangle$ such that $\lambda_x(\langle x, y\rangle) = a$, $\lambda_y(\langle y, x\rangle) = b$, $\lambda_y(\langle y, z\rangle) = a$, $\lambda_z(\langle z, y\rangle) = c$, with $b \neq c$. By definition of a consistent coding function, we have that $f(\Lambda_x([\langle x, y\rangle, \langle y, x\rangle, \langle x, y\rangle]))$ $= f(\lambda_x(\langle x, y\rangle))$, that is $f([a, b, a]) = f(a)$, we also have that $f(\Lambda_y([\ \langle y, z\rangle,\ \langle z, y\rangle,\ \langle y, z\rangle]))$ $= f(\lambda_y(\langle y, z\rangle))$, that is $f([a, c, a]) = f(a)$.
There are two possible situations depending on whether $a \neq c$ or $a = c$.

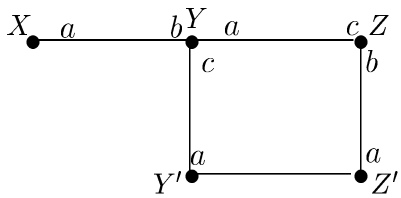

Figure 3: Proof of Lemma 1, case $a \neq c$.

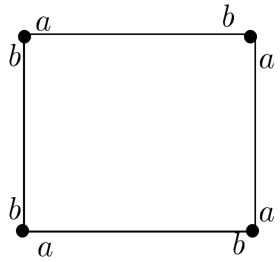

Figure 4: Proof of Lemma 1, case $a = c$.

Let $a \neq c$. Since the graph is 3-regular and uses three labels, there must exist a path labeled $[a, c, a]$ starting from any vertex. Let $[\langle x, y\rangle, \langle y, y'\rangle, \langle y', y''\rangle]$ be such a path starting from x. Since $f([a, c, a]) = f(a)$, it follows that $y'' = y$ (see Figure 3). Using the same reasoning, we can show that there exists an edge $\langle z, z'\rangle$ such that $\lambda_z(\langle z, z'\rangle) = b$, and $\lambda_{z'}(\langle z', z\rangle) = a$. Consider now the path $\pi = [\langle y, z\rangle, \langle z, y\rangle, \langle y, y'\rangle]$. By definition of consistent coding function we have that $f(\Lambda_y(\pi)) = f(\lambda_y(\langle y, y'\rangle))$, that is $f([a, c, c]) = f(c)$; we also have $f(\Lambda_{z'}([\ \langle z', z\rangle,$ $\langle z, y\rangle,\ \langle y, y'\rangle)) = f(a, c, c)$. Thus, there must exists an edge $\langle z', y'\rangle$ such that $\lambda'_z(\langle z', y'\rangle) = c$. But this means that there exists a cycle $[y, z, z', y']$ of length 4, which is impossible since Petersen's graph does not have any cycle of length 4.

Let now $a = c$. In this case, it is easy to see that the graph must contain the cycle of length 4 shown in Figure 4. But this is impossible since Petersen's graph does not have any cycle of length 4. □

Theorem 3 Cycle symmetricity *is not sufficient for minimal $\mathcal{SD}$ in regular graphs.*

Proof Consider Petersen's graph (see Figure 2); this graph is vertex symmetric and, thus, by Lemma 1 is cyle symmetric, nevertheless, Petersen's graph has no minimal $\mathcal{SD}$.
By contradiction, suppose there exists a minimal Sense of Direction; obviously, it would use three labels. By Theorem 1 the labeling must have edge consistency. It is easy to see that any such a labeling is isomorphic to the one in Figure 2. Consider the labeling of Figure 2. Consider the two paths $\pi_1, \pi_2 \in P[A, G]$ such that $\Lambda_A(\pi_1) = [1, 3, 2]$ and $\Lambda_A(\pi_2) = [2, 3]$. By definition of consistent coding function we have that

$$f([1, 3, 2]) = f([2, 3]) \tag{3}$$

Consider now the two paths $\pi_3 \in P[L, D], \pi_4 \in P[L, B]$ such that $\Lambda_L(\pi_3) = [1, 3, 2]$ and $\Lambda_L(\pi_4) = [2, 3]$. By definition of consistent coding function we have that

$$f([1, 3, 2]) \neq f([2, 3]) \tag{4}$$

which contradicts (3). $\square$

4 Cayley Graphs

In this Section we consider a particular class of regular graphs, Cayley graphs, and we show that their natural labeling is always a minimal Sense of Direction.

Given a set of generators S for a finite group $\mathcal{G}$, a Cayley graph is a graph $N_{\mathcal{G}} = (V, E)$, where the vertices correspond to the elements of the group ($V = \mathcal{G}$) and the edges correspond to the action of the generators; that is $\langle x, y \rangle \in E$ iff $\exists g \in S : x \circ g = y$, where $\circ$ is the usual composition of functions. The set of generators is closed under inverses so that the graph is undirected.

Cayley graphs have many properties [1, 5, 7, 16, 17, 18, 27], which make them very interesting as models for interconnection networks.

It is well known that every Cayley graph is vertex symmetric, thus, by Theorem 1, it follows that:

Corollary 1 *Every Cayley graph is cycle symmetric.*

Let $\mathcal{L} = \mathcal{S}$; the natural labeling λ for a Cayley graph $N_{\mathcal{G}}$ is the following:

Definition 10 Cayley Labeling
Given a system $(N_{\mathcal{G}}, \lambda)$, where $N_{\mathcal{G}}$ is a Cayley graph, λ is a Cayley *labeling iff* $\forall \langle x, y \rangle \in E(x)$*:* $\lambda_x(\langle x, y \rangle) = g$*, where g is the generator such that $y = x \circ g$.*

We give the following definition,

Definition 11 *A coding function f for a Cayley labeling is* associative *iff:* $\forall \langle x, y \rangle \in E(x)$, $\forall \pi \in P[y]$

$$f([\lambda_x(\langle x, y \rangle) \circ f(\Lambda_y(\pi))]) = f([\lambda_x(\langle x, y \rangle) \circ \Lambda_y(\pi)])$$

where $\circ$ is the usual composition of functions.

Similarily to [9], we can easily prove that: given a system (G, λ, β), where λ is Cayley labeling,

Lemma 2 *If there exists an associative consistent coding function f, then there exists a consistent decoding function h for f, thus, λ is a Sense of Direction.*

The following Theorem shows that any Cayley labeling is globally consistent, that is, there exists a naming function such that the labeling is a Sense of Direction.

Theorem 4 *Any Cayley labeling λ of a Cayley graph $N_{\mathcal{G}}$ is globally consistent.*

Proof Let $\mathcal{N} = \mathcal{G}$ and $\mathcal{L} = \mathcal{S}$. To verify that λ is globally consistent, consider the coding function f defined as follows: $\forall \pi \in P[x]$, with $\Lambda_x(\pi) = [g_1, \ldots g_k]$

$$f(\Lambda_{x_0}(\pi)) = g_1 \circ g_2 \circ \ldots \circ g_k$$

For any $x, y, z \in V$, consider any two paths $\pi_1 \in P[x,y]$ with $\Lambda_x(\pi_1) = [g_1, \ldots g_k]$ and $\pi_2 \in P[x,z]$ with $\Lambda_x(\pi_2) = [s_1, \ldots s_h]$. We have to show that $f(\Lambda_x(\pi_1)) = f(\Lambda_x(\pi_2))$ iff $y = z$.

a) We first show that $f(\Lambda_x(\pi_1)) = f(\Lambda_x(\pi_2)) \Rightarrow y = z$.
Suppose $f(\Lambda_x(\pi_1)) = f(\Lambda_x(\pi_2))$. By definition of consistent coding function, we have that $g_1 \circ \ldots \circ g_k = s_1 \circ \ldots \circ s_h$, thus, $x \circ g_1 \circ \ldots \circ g_k = x \circ s_1 \circ \ldots \circ s_h$, and, thus, $y = z$.
b) We now show that $y = z \Rightarrow f(\Lambda_x(\pi_1)) = f(\Lambda_x(\pi_2))$.
Suppose that $y = z$. By definition of Cayley labeling, we have that $x \circ g_1 \circ \ldots \circ g_k = x \circ s_1 \circ \ldots \circ s_h$, thus, $g_1 \circ \ldots \circ g_k = s_1 \circ \ldots \circ s_h$ and, thus, $f(\Lambda_x(\pi_1)) = f(\Lambda_x(\pi_2))$.

It follows that, if we choose $\beta_x(y) = f(\alpha)$ where α is any sequence of labels corresponding to a path between x and y we have that $\beta_x(y) = t$ such that $x \circ t = y$ and f is consistent.

We will now show that the consistent coding function f is associative. First of all, observe that $\mathcal{N} = \mathcal{L}$. For any path $\pi \in P[y,z]$ and any edge $\langle x, y \rangle$, let $\Lambda_y(\pi) = [g_1, \ldots, g_m]$, $\lambda_x(\langle x, y \rangle) = g_0$, and consider $f([\lambda_x(\langle x, y \rangle) \circ f(\Lambda_y(\pi)])$. By definition of f, $f(\Lambda_y(\pi) = [g_1 \circ g_2 \ldots \circ g_m]$, thus, $f([\lambda_x(\langle x, y \rangle) \circ f(\Lambda_y(\pi)]) = [g_0 \circ g_1 \ldots \circ g_m] = f(\lambda_x(\langle x, y \rangle) \circ \Lambda_y(\pi))$. Thus, f is associative.

By Lemma 2 it follows that there exists a consistent decoding function h of f and, thus, λ is globally consistent. □

In other words, given a Cayley Graph G labeled with the natural labeling λ, there exists a consistent coding, a consistent decoding, and, thus, a naming function β such that λ is a $\mathcal{SD}$ in (G, λ, β). A Cayley labeling λ is called *Cayley* $\mathcal{SD}$ in (G, λ, β) when β is defined as in the proof of the previous Theorem. An example of Cayley $\mathcal{SD}$ is given in Figure 5.

Corollary 2 *For every Cayley graph G there exists a minimal Sense of Direction.*

Proof By definition, a Cayley graph is a d-regular graph G whose natural labeling λ uses exactly d labels. By Theorem 4 we have that there exists a naming function β such that λ is a $\mathcal{SD}$ in (G, λ, β). It follows that the natural $\mathcal{SD}$ is minimal. □

When a $\mathcal{SD}$ λ has edge consistency and the corresponding naming function β has name consistency, λ is a stronger form of Sense of Direction and it is called *Symmetric Sense of Direction.*

Definition 12 $\mathcal{SSD}$ - Symmetric Sense of Direction
Given (G, λ, β), λ is a Symmetric Sense of Direction *if λ is a $\mathcal{SD}$ with edge consistency and β has name consistency.*

The properties of Symmetric $\mathcal{SD}$ have been studied in [10]. In particular, it has been shown that the availability of Symmetric Sense of Direction gives to the system strong translation capabilities and that some of the instances of $\mathcal{SD}$ which have been used in the literature are Symmetric.

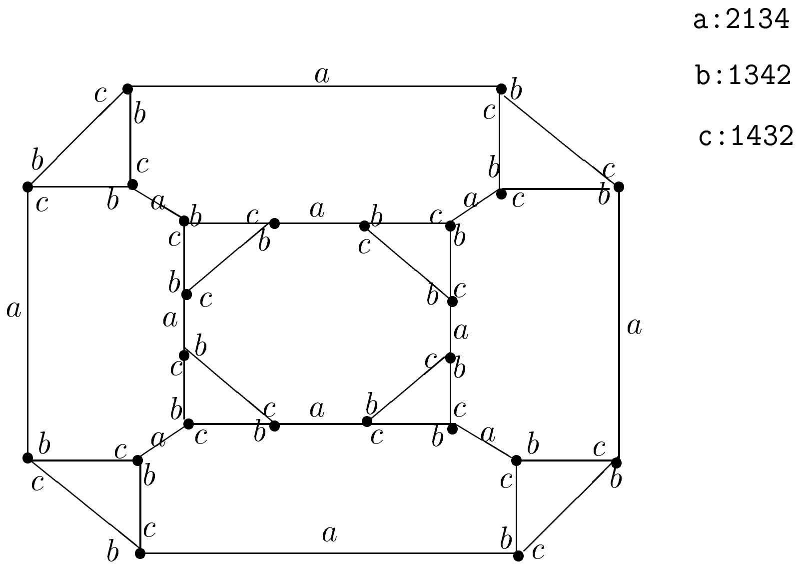

Figure 5: A Cayley graph with a $\mathcal{SD}$.

It is easy to prove that a Cayley $\mathcal{SD}$ is Symmetric.

Theorem 5 *A Cayley $\mathcal{SD}$ λ is Symmetric.*

Proof Let λ be a Cayley $\mathcal{SD}$ in (G, λ, β). We have to show that λ and β have edge and name consistency, respectively. Consider the function $\psi : \mathcal{L} \to \mathcal{L}$, $\psi(g) = g^{-1}$, where g^{-1} is such that $g \circ g^{-1}$ is the identity function. Function ψ is an edge consistency function. In fact, let x and y be neighbors, then, by definition of Cayley labeling $\lambda_x(\langle x, y\rangle) = g$ such that $x \circ g = y$, and $\lambda_y(\langle y, x\rangle) = q$ such that $y \circ q = x$. It follows that $q = g^{-1}$. Thus, ψ is the edge consistency function. Consider the function $\mu : \mathcal{N} \to \mathcal{N}$, $\psi(t) = t^{-1}$, where t^{-1} is such that $t \circ t^{-1}$ is the identity function. By definition of naming function, we have that $\beta_u(v) = t$ such that $u \circ t = v$ and $\beta_v(u) = z$ such that $v \circ z = u$, it follows that μ is a name consistency function; thus, β has also name consistency and λ is a Symmetric Sense of Direction. $\square$

Acknowledgements. The author would like to thank Nicola Santoro for the many and insightful discussions and Frederic Geurts for providing helpful comments.

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