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TOPOLOGICAL CONSTRAINTS FOR SENSE OF DIRECTION*

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ABSTRACT

In a distributed system, each entity has a label (*port number*) associated to each of its neighbors. It is well known that if the labeling satisfies a global consistency property called *Sense of Direction*, the communication complexity of many distributed problem can be greatly reduced. This property is assumed to be present and it is implicitly used in most interconnection networks.
In this paper we investigate the nature of the relationship between Sense of Direction and network topology. We first consider the class of labelings which only satisfy the requirement of *local orientation*; that is, at each node, the outgoing edges have distinct labels. We then consider the class of labelings which also satisfy *edge symmetry*; that is, there exists a correlation between the labels neighbors give to their connecting edge. This class includes most of used labelings. We completely characterize the interplay between topological constraints and consistency constraints for these two classes of labelings. In fact, for each class, we identify both the set of *forbidden subgraphs* and the set of *feasible graphs*, and we show that the characterization is complete.

Keywords: Sense of Direction, Network Topology, Distributed Computing.

1. Introduction

A *distributed system* is a collection of processing entities (e.g., processors) which communicate by exchanging finite sequences of bits (i.e., messages). The *communication topology* of the system is a graph $G(V, E)$ where nodes correspond to processing entities, and edges correspond to direct bidirectional communication links between

*A preliminary version of this paper has been presented to the 2nd Colloquium on Structural Information and Communication Complexity entities. Every entity has a local label (also called *port number*) associated with each of its incident links. Let $\lambda_x(\langle x, z\rangle)$ denote the label associated by x to the incident link $\langle x, z\rangle$. Each entity x refers to the other entities using local labels called *names*. Let $\beta_x(y)$ be the name associated by x to y. These local names are *not* necessarily identities (i.e., unique global identifiers); in fact, the system could be anonymous.

The entire system is thus described by the triple (G, λ, β), where $\lambda = \{\lambda_x : x \in V\}$ is the *labeling function* describing the association of port numbers to edges, and $\beta = \{\beta_x : x \in V\}$ is the *naming function* describing the association of local names to nodes.

Some assignments of port numbers (i.e., some labelings λ) have properties which can be exploited in the solution of distributed problems (e.g., routing, election, traversal). Of particular interest is the property called *Sense of Direction* ($\mathcal{SD}$) [1,2,3]. It is well known that the presence of Sense of Direction can have a dramatic effect on the performance of communication protocols for a large class of distributed problems including *election* and *spanning-tree construction* (e.g., see [4,5,6,7,8,9,10,11,12,13]), *depth-first traversal* [14,15], *distributed function evaluation* (e.g., see [16,17,18,19,20,21]), *clock synchronization* [22], etc.

Observe that almost all the labelings considered in the studies of interconnection networks are $\mathcal{SD}$s. In other words, the existing results (e.g., [23,24,25,26,27]) show that Sense of Direction has an immediate impact on most communication problems in interconnection networks: e.g., *broadcasting*, *gossiping*, *routing*, *permuting*, etc.

Similarly, in redundant ring networks (e.g., [28,29,31]) some results on *reconfiguration* implicitly exploit the fact that the labeling has Sense of Direction [32,33].

Because of this impact on complexity, the characterization and the analysis of the properties of Sense of Direction are clearly important, from both theoretical and practical point of views.

Most of the existing research has focused on the study of specific labelings in specific topologies: the "left-right" labeling of rings (e.g., [34,35,36,37]), *chordal* labelings in complete networks and chordal rings (e.g., [38,10]), *dimensional* labelings in hypercubes (e.g., [4,11]), *compass* labelings in meshes and tori (e.g., [16,39,40]), *contracted* labeling in most interconnection networks, etc.

Questions have also been posed at a more general level, focusing e.g. on the study of the computational capabilities of $\mathcal{SD}$ [41], and on the analysis of the complexity of determining whether a labeling is a $\mathcal{SD}$ [42].

The set of global consistency constraints which define Sense of Direction simultaneously link both "semantic" and "topological" aspects. There is no doubt that the network topology has an immediate and crucial impact in determining when the "semantic" constraints can be satisfied. On the other hand, very little is known on the nature of this impact.

In this paper, we investigate the nature of the strong interplay existing between the structure of the system (i.e., topology of the graph G) and the consistency properties which a labeling must satisfy to be a $\mathcal{SD}$. In particular, we are interested in the following question:

Given a labeling property $\mathcal{P}$, under what topological conditions the labelings satisfying $\mathcal{P}$ will always be globally consistent? In other words, *for which graphs, $\mathcal{P}$ implies Sense of Direction?* and *for which graphs $\mathcal{SD}$ is not implied by $\mathcal{P}$?*

In this paper, we provide a complete answer for two large classes of labelings, *Local Orientations* ($\mathcal{LO}$), and *Locally Symmetric Orientations* ($\mathcal{LSO}$).

Local orientations are labelings where, at each node, the edges incident to it have distinct labels. This property is the minimal requirement for $\mathcal{SD}$ as well as for most computations.

Locally symmetric orientations $\mathcal{LSO}$ are those local orientations which satisfy also the *edge symmetry* constraint: there exists a correlation between the labels neighbors give to their connecting edge. Observe that most labeling are locally symmetric orientations; e.g., the "left-right" labeling of rings, the *chordal* labelings in complete networks and chordal rings, the *dimensional* labelings in hypercubes, *compass* labelings in meshes and tori, *contracted* labeling, etc.

For each class of labeling, we identify and characterize both the set of *forbidden subgraphs* and the set of *feasible graphs*.

This characterization is *complete*: as we show, every graph is either feasible or contains a forbidden subgraph.

Not surprisingly, local orientation does not suffice for Sense of Direction except for trivial graphs. On the other hand, the simultaneous presence of local orientation and edge symmetry guarantees Sense of Direction in a large class of graphs, most interconnection networks, etc.

In our results, an important role is played by a special class of graphs, the *spiked rings* which appear in the characterization of both forbidden and feasible graphs.

These results formally characterize for the first time the nature of the interplay between network topology and Sense of Direction for $\mathcal{LO}$, and $\mathcal{LSO}$ labelings.

In view of the impact of $\mathcal{SD}$ on complexity and of the generality of these classes of labelings, these results constitute a solid step towards the understanding of the nature of the relationship between distributed computing and interconnection networks.

2. Definitions

2.1. Orientations

The communication topology of a distributed system is a connected graph $G(V, E)$ where nodes correspond to processing entities and edges correspond to direct bidirectional communication links between entities. Given $x \in V$, let $E(x) \subseteq E$ denote the set of edges incident to x. A *walk* in G is a sequence of edges $[\langle x_0, x_1\rangle, \langle x_1, x_2\rangle, \ldots, \langle x_{m-1}, x_m\rangle]$, $\langle x_i, x_{i+1}\rangle \in E(x_i)$, in which the endpoint of one edge is the starting point of the next edge. In the following, a walk will be denoted also as the ordered sequences of the nodes $[x_0, x_1, \ldots, x_m]$, when no ambiguity arises. A walk is a *path* if $x_i \neq x_j$ for $i \neq j$, and it is a *cycle* if $x_0 = x_m$. Let $P[x]$ denote the set of all the walks with $x \in V$ as a starting point, and let $P[x, y]$ denote the set of walks starting from node $x \in V$ and ending in node $y \in V$.

Every node $x \in V$ has a label (also called *port number*), from a set of labels $\mathcal{L}$, associated to each of its incident edges; let $\lambda_x(\langle x, z \rangle)$ denote the label associated by x to $\langle x, z \rangle \in E(x)$. Let $\lambda = \{\lambda_x : x \in V\}$ be the *labeling function* and let Λ be its extension from edges to walks, that is: $\Lambda(\ [\langle x_0, x_1 \rangle,\ \ldots,\ \langle x_{m-1}, x_m \rangle]\ ) = [\lambda_{x_0}(\langle x_0, x_1 \rangle),\ \ldots,\ \lambda_{x_{m-1}}(\langle x_{m-1}, x_m \rangle)]$. Let $\circ$ denote the concatenation of strings.

Each node x refers to the other nodes using local labels, called *names*, from a set $\mathcal{N}$; let $\beta_x(y)$ be the name associated by x to y. These local names are *not* necessarily identities (i.e., unique global identifiers); in fact, the system could be anonymous.

The entire system can thus be described as the triple (G, λ, β), where $\lambda = \{\lambda_x : x \in V\}$ is the *labeling function* and $\beta = \{\beta_x : x \in V\}$ is the *naming function*.

A basic computational requirement in a system (G, λ, β) is that each entity has *local orientation*; that is, it can distinguish among its neighboring entities. In other words:

Definition 1 $\mathcal{LO}$ - Local Orientation
A labeling λ *is a* Local Orientation *iff* $\forall x \in V$, $\forall e_1, e_2 \in E(x)$,

$$\lambda_x(e_1) = \lambda_x(e_2) \quad \text{iff} \quad e_1 = e_2$$

In most systems, there exists a predetermined relation between the labels at the two side of an edge; e.g., the "left" and "right" labels in a ring. A labeling has *edge symmetry* iff there exists a function $\psi : \mathcal{L} \to \mathcal{L}$, such that, $\forall \langle x, y \rangle \in E$, $\lambda_x(\langle x, y \rangle) = \psi(\lambda_y(\langle y, x \rangle))$. Let $\Psi : \mathcal{L}^+ \to \mathcal{L}^+$ be the extension of the edge symmetry function from edges to walks.

Definition 2 $\mathcal{LSO}$ - Locally Symmetric Orientation
A labeling is a Locally Symmetric Orientation *when it is a local orientation with edge symmetry.*

Let us observe that most of the labelings studied in the literature are $\mathcal{LSO}$.

2.2. Sense of Direction and Global Consistency

A *coding function* f in (G, λ, β) is a function which maps to names the sequences of labels associated to walks in G. A coding function f is *consistent* if $\forall x, y \in V$, $\pi \in P[x, y]$, $f(\Lambda_x(\pi)) = \beta_x(y)$; in other words, it maps all the (sequences of labels associated to) walks from x to y into the name that x associates to y. By definition of β, this implies that $\forall \pi_1 \in P[x, y]$, $\pi_2 \in P[x, z]$, $f(\Lambda_x(\pi_1)) = f(\Lambda_x(\pi_2))$ iff $y = z$.

A *decoding function* h is a function which associates a name to a given name and a label. Given a consistent coding function f, a decoding function h is *consistent* for f if $\forall \langle x, y \rangle \in E(x), \forall \pi \in P[y, z]$ $h(\lambda_x(\langle x, y \rangle), f(\Lambda_y(\pi))) = \beta_x(z)$. In other words, a consistent decoding function allows a node to "translate" names that neighbors associate to nodes into its own local names for those nodes.

Definition 3 $\mathcal{SD}$ - Sense of Direction
Given (G, λ, β), λ *is a* Sense of Direction *if the following conditions hold:*
1) There exists a consistent coding function f.
2) There exists a consistent decoding function h *for* f.

Notice that $\mathcal{SD}$ requires local orientation.

Definition 4 Inconsistency and Global Consistency
Given a labeled graph (G, λ), λ *is* globally consistent *if there exists a* β *such that* λ *is a* $\mathcal{SD}$ *in* (G, λ, β)*; otherwise,* λ *is said to be* inconsistent*.*

In other words, λ is inconsistent if λ is *not* a $\mathcal{SD}$ in (G, λ, β) for any choice of β.

2.3. Spiked Rings

In this Section we introduce a class of graphs, here termed *Spiked Ring*, which will be used in the following sections.

Given a rooted tree T, let $r(T)$ denote the root of T and $h(T)$ denote the height of T.

Definition 5 Spiked Rings
Let $T^1, T^2, \ldots, T^n$ *be non-empty rooted trees. A* spiked ring of size n with spike structure $\{T^1, \ldots, T^n\}$ *is the graph, denoted by* $R_n\langle T^1, \ldots, T^n\rangle$*, obtained by connecting each root* $r(T^i)$ *with roots* $r(T^{i-1})$ *and* $r(T^{i+1})$*, where the operations on the indices are modulo* n*. Each tree* T^i *will be called a spike.*

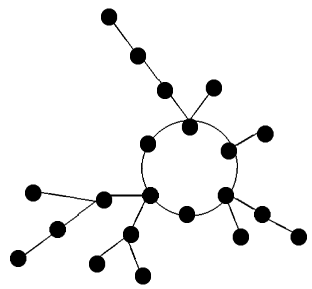


Figure 1: A spiked ring.

A class of spiked rings of particular interest in this paper is the one of the *hat*. Let L_n denote a chain of size n.

Definition 6 Hat Rings
A spiked ring $R_n\langle T^1, \ldots, T^n\rangle$ *is said to be a* (p, q)-hat ring*, denoted by* $\widehat{R_n}(p, q)$*, if* $\exists i, j$ *such that* $T^i = L_p$*,* $T^j = L_q$*,* $(j - i)\mathrm{mod} n = p + q$ *and* $T_k = L_0$ $\forall i \neq k \neq j$*.*

That is, a (p, q)-hat ring is a ring with two spikes, which are chains, one of length p and the other of length q, at distance $p + q$ from each other.

Some particular spiked rings which will be used in the following are: the *spider* and the *hat-free spider*.

Definition 7 Spider
A spiked ring $R_n\langle T^1, \ldots, T^n\rangle$ *is said to be a* spider*, denoted by* S_n*, if* $\forall i$*,* $h(T^i) = 1$*.*

Definition 8 Hat-free spider
A spider S_n *is said to be a* hat-free spider*, denoted by* $\overline{S}_n$*, if it does not contain any (1,1)-hat ring.*

In the following we shall denote by R_n a ring of size n, by K_n a complete graph of size n, by T_n a tree of size n, and by $R_{n;h}$ a ring of size n with h bisecting chords.

3. Constraints for $\mathcal{LO}$

In this section we consider labelings which satisfy only the constraint of local orientation, that is, all edges incident to the same node have different local labels. We show that local orientation is sufficient for $\mathcal{SD}$ only in trivial graph.

Lemma 1 *Given a labeled graph (G', λ'), where $G' \subseteq G$, and λ' is a local orientation, there exists a completion λ of λ' in G, such that λ is a local orientation in G.*

Proof. Trivial, e.g., by assigning distinct labels to all edges in $G - G'$. □

Theorem 1 *Local orientation is not sufficient for $\mathcal{SD}$ in R_3, nor in any graph with $n > 3$ nodes.*

Proof. We first consider the case $G = R_3$.
Case a). Let $[x, y, z]$ be the nodes of R_3. Let a, b be two distinct labels in $\mathcal{L}$. Label the edges $\langle x, y\rangle$, $\langle y, x\rangle$, and $\langle z, x\rangle$ with a; and label $\langle y, z\rangle$, $\langle z, y\rangle$ and $\langle x, z\rangle$ with b. (see Figure 2). By contradiction, suppose this labeling is a $\mathcal{SD}$ and let f be the corresponding coding function. By definition of consistent coding function, $f(\Lambda_x(\langle x, y\rangle, \langle y, z\rangle)) = f(\Lambda_x(\langle x, z\rangle))$; that is, $f([a, b]) = f(b)$. However, also by definition of consistent coding function, $f(\Lambda_z(\langle z, x\rangle, \langle x, z\rangle)) \neq f(\Lambda_z(\langle z, y\rangle))$; that is, $f([a, b]) \neq f(b)$ which yields the contradiction.

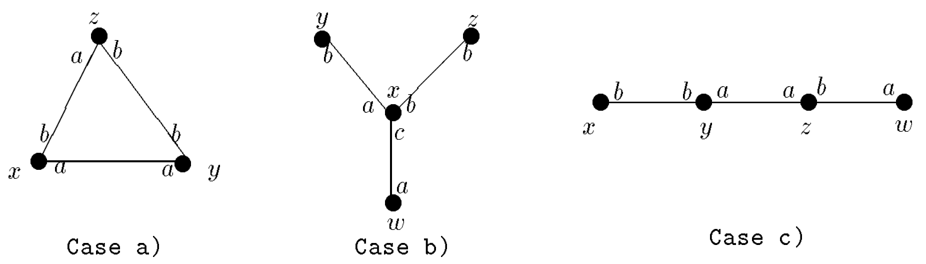


Figure 2: Proof of Theorem 1.

Consider now the case when G has $n > 3$ nodes. We will consider two sub-cases, depending on whether or not there exists in G a node of degree 3.
Case b): There exists a node of degree at least 3.
Let x be this node, and let y, z, w be its neighbors. Let a, b, c be three distinct labels in $\mathcal{L}$. Label the edges $\langle x, y\rangle$ and $\langle w, x\rangle$ with a; $\langle y, x\rangle$, $\langle x, z\rangle$ and $\langle z, x\rangle$ with b; and $\langle x, w\rangle$ with c (see Figure 2). Complete the labeling of G maintaining the property of local orientation (this is always possible by Lemma 1). By contradiction, suppose this labeling is a $\mathcal{SD}$ and let f be the corresponding coding function. By definition of consistent coding function, $f(\Lambda_x(\langle x, y\rangle, \langle y, x\rangle, \langle x, z\rangle)) = f(\Lambda_x(\langle x, z\rangle))$; that is,

$f([a,b,b]) = f(b)$ (see Figure 2). However, also by definition of consistent coding function, $f(\Lambda_w(\langle w,x\rangle,\langle x,z\rangle,\langle z,x\rangle)) = f(\Lambda_w(\langle w,x\rangle))$; that is, $f([a,b,b]) = f(a)$. It follows that $f(a) = f(b)$. Consider the local names that x associates to y and z; $\beta_x(y) = f(a)$ and $\beta_x(z) = f(b)$, thus, $\beta_x(y) = \beta_x(z)$ which contradicts the definition of naming function.
Case c): There are no nodes of degree 3
In this case, there must exist in G a path of length $p > 2$. Let $x,y,z,w \in V$ be consecutive nodes on such a path. Let a,b,c be three distinct labels in $\mathcal{L}$. Label the edges $\langle y,z\rangle$, $\langle z,w\rangle$, and $\langle w,z\rangle$ with a; $\langle x,y\rangle$, $\langle y,x\rangle$, and $\langle z,w\rangle$ with b (see Figure 2). Complete the labeling of G maintaining the property of a local orientation (this is always possible by Lemma 1). By contradiction, suppose this labeling is a $\mathcal{SD}$ and let f be the corresponding coding function. By definition of consistent coding function, it must be $f(\Lambda_z(\langle z,w\rangle,\langle w,z\rangle,\langle z,w\rangle)) = f(\Lambda_z(\langle z,w\rangle))$; that is, $f([b,a,b]) = f(b)$ (see Figure 2) However, also by definition of consistent coding function, $f(\Lambda_x(\langle x,y\rangle,\langle y,z\rangle,\langle z,w\rangle)) = \beta_x(w)$; that is, $f([b,a,b]) = \beta_x(w)$ and $f(\Lambda_x(\langle x,y\rangle)) = \beta_x(y)$; that is, $f(b) = \beta_x(y)$. It follows that $\beta_x(w) = \beta_x(y)$ which yields the contradiction. □
It is easy to see that
Theorem 2 *Local orientation is sufficient for $\mathcal{SD}$ in L_n, with $n \le 3$.*
From Theorems 1 and 2, it follows that
Theorem 3 *Every local orientation λ of G is globally consistent iff $G = L_n$, with $n \le 3$.*

4. Forbidden Subgraphs for $\mathcal{LSO}$

In this section we consider the class of local orientations which satisfy also the edge symmetry constraint; that is, on each edge, the label at one side is a function of the label at the other side (*Locally Symmetric Orientation*). We characterize the class of graphs in which locally symmetric orientation is not sufficient for $\mathcal{SD}$, by showing a set of *forbidden subgraphs.*

The following lemma shows a structural condition under which a given labeling is inconsistent.

Lemma 2 *Let λ have edge symmetry in (G,λ). If there exist $x,y,z \in V$, $y \ne z$ and $\pi_1 \in P[x,x]$, $\pi_2 \in P[y,z]$ such that*

$$\Lambda_x(\pi_1) = \Lambda_y(\pi_2)$$

then λ is inconsistent in G.

Proof. Let λ have edge symmetry in (G,λ), and let $x,y,z \in V$, $y \ne z$ and $\pi_1 \in P[x,x]$, $\pi_2 \in P[y,z]$ such that $\Lambda_x(\pi_1) = \Lambda_y(\pi_2)$.
By contradiction, let β denote a naming function such that λ is a $\mathcal{SD}$ in (G,λ,β). Let f be the corresponding consistent coding function.

Consider an arbitrary node $w \in \pi_1$. It divides π_1 in two sub-walks $\pi_{1,1} \in P[x,w]$ and $\pi_{1,2} \in P[w,x]$ (such that $\pi_1 = \pi_{1,1} \circ \pi_{1,2}$). Let $\alpha_1 = \Lambda_x(\pi_{1,1})$ and $\alpha_2 = \Lambda_w(\pi_{1,2})$. Since $\Lambda_x(\pi_1) = \Lambda_y(\pi_2)$, there exists a node $w' \in \pi_2$ which divides π_2 in two sub-walks $\pi_{2,1} \in P[y,w]$ and $\pi_{2,2} \in P[w,z]$ such that $\alpha_1 = \Lambda_y(\pi_{2,1})$ and $\alpha_2 = \Lambda_w(\pi_{2,2})$.

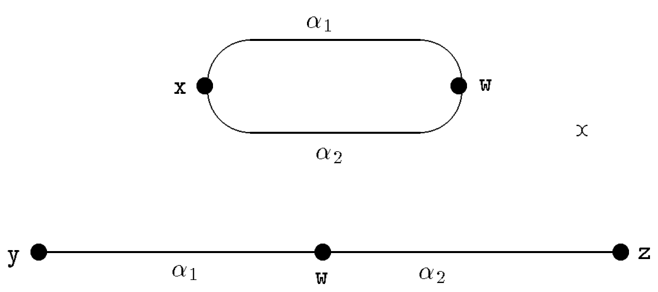


Figure 3: Proof of Lemma 2.

By definition of Sense of Direction and of edge symmetry function,

$$f(\Psi(\Lambda_w(\pi_{1,1}))) = f(\Lambda_w(\pi_{1,2}))$$

that is,

$$f(\alpha_2) = f(\Psi(\alpha_1))$$

where ψ and Ψ denote the edge and walk symmetry function, respectively. However, by definition of Sense of Direction and of edge symmetry function,

$$f(\Psi(\Lambda_{w'}(\pi_{2,1}))) \neq f(\Lambda_{w'}(\pi_{2,2}))$$

that is,

$$f(\alpha_2) \neq f(\Psi(\alpha_1))$$

which yields the contradiction. □

In other words, there exist locally symmetric orientations for which there is no naming function which would make the labeling a Sense of Direction.

Lemma 3 *Given a labeled graph (G', λ'), where $G' \subseteq G$, and λ' is a locally symmetric orientation, there exists a completion λ of λ' in G, such that λ is a locally symmetric orientation.*

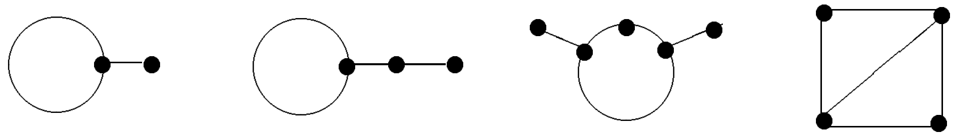
Proof. Trivial, e.g., by assigning distinct labels to all edges in $G - G'$. □

Theorem 4 *Locally symmetric orientation is not sufficient for $\mathcal{SD}$ in a graph containing any of the following subgraphs:*
a) a $(1,0)$-hat ring $\widehat{R_{2n}}(1,0)$;
b) a $(2,0)$-hat ring $\widehat{R_{2n+1}}(2,0)$;
c) a $(1,1)$-hat ring $\widehat{R_{2n+1}}(1,1)$;
d) a ring of size four with one chord $R_{4;1}$.

Proof. By contradiction, let G contain one of the above forbidden subgraphs and let any $\mathcal{LSO}$ in G be a $\mathcal{SD}$. We will consider the four cases separately.

Case a): G contains a $(1,0)$-hat ring of even size.
In $\widehat{R_{2n}}(1,0)$, let $[y_0, y_1 \ldots y_m]$ be the nodes of the ring, $m = 2n-1$, and let $\langle x, y_0 \rangle$

be the spike. Let a, b be two distinct labels. Label the edges $\langle x, y_0\rangle$ and $\langle y_0, y_1\rangle$ with a; and label $\langle y_1, y_0\rangle$ and $\langle y_0, x\rangle$ with b. Label the path $[\langle y_1, y_2\rangle, \ldots, \langle y_m, y_0\rangle]$ with a palindrome string $\alpha \in \mathcal{L}^{2n-1}$; i.e. $\Lambda_{y_1}([\langle y_1, y_2\rangle, \ldots, \langle y_m, y_0\rangle]) = \Lambda_{y_0}([\langle y_0, y_m\rangle, \langle y_m, y_{m-1} \rangle, \ldots, \langle y_2, y_1\rangle]) = \alpha$ (this is possible, since the length of the path is odd). This labeling of $\widehat{R_{2n}}(1,0)$ is clearly a $\mathcal{LSO}$. Complete the labeling of G maintaining the property of locally symmetric orientation (this is possible by Lemma 3).
Consider the path $\pi_1 = [x, y_0, y_m, y_{m-1}, \ldots y_1] \in P[x, y_1]$ and the cycle $\pi_2 = [y_0, y_1, \ldots, y_m, y_0] \in P[y_0, y_0]$; we have that $\Lambda_x(\pi_1) = [a \circ \alpha] = \Lambda_{y_0}(\pi_2)$.
By Lemma 2 it follows that λ is inconsistent, yielding the contradiction.



$\widehat{R_{2n}}(1,0)$ $\widehat{R_{2n+1}}(2,0)$ $\widehat{R_{2n+1}}(1,1)$ $R_{4;1}$

Figure 4: LSO-forbidden graphs

Case b): G contains a $(2,0)$-hat ring of odd size.
In $\widehat{R_{2n+1}}(2,0)$, let $[y_0, y_1 \ldots y_m]$ be the nodes of the ring, where $m = 2n$, and let $[x_0, x_1, y_0]$ be the nodes of the spike. Let a, b be two distinct labels. Label the edges $\langle x_0, x_1\rangle$ and $\langle y_0, y_1\rangle$ with a; and label $\langle x_1, y_0\rangle$ and $\langle y_1, y_2\rangle$ with b. Label the path $[\langle y_2, y_3 \rangle, \ldots, \langle y_m, y_0 \rangle]$ with a palindrome string $\alpha \in \mathcal{L}^{2n-1}$, i.e., $\Lambda_{y_2}([\langle y_2, y_3\rangle, \ldots, \langle y_m, y_0\rangle]) = \Lambda_{y_0}([\langle y_0, y_m\rangle, \langle y_m, y_{m-1}\rangle, \ldots, \langle y_3, y_2\rangle]) = \alpha$ (this is possible, since the length of the path is odd). This labeling of $\widehat{R_{2n}}(2,0)$ is clearly a $\mathcal{LSO}$. Complete the labeling of G maintaining the property of $\mathcal{LSO}$ (this is possible by Lemma 3).
Consider the path $\pi_1 = [x_0, x_1, y_0, y_m, y_{m-1}, \ldots y_2] \in P[x_0, y_2]$ and the cycle $\pi_2 = [y_0, y_1, \ldots, y_m, y_0] \in P[y_0, y_0]$; we have that $\Lambda_x(\pi_1) = [a \circ b \circ \alpha] = \Lambda_{y_0}(\pi_2)$.
By Lemma 2 it follows that λ is inconsistent, yielding the contradiction.

Case c): G contains a $(1,1)$-hat ring of odd size.
In $\widehat{R_{2n+1}}(1,1)$, let $[y_0, y_1 \ldots y_m]$ be the nodes of the ring, where $m = 2n$, and let $\langle u, y_0\rangle$ and $\langle v, y_2\rangle$ be the two spikes of length one, at distance two from each other. Let a, b be two distinct labels. Label the edges $\langle u, y_0\rangle$ and $\langle y_1, y_2\rangle$ with a; and label $\langle v, y_2\rangle$ and $\langle y_0, y_1\rangle$ with b. Label the path $[\langle y_2, y_3\rangle, \ldots, \langle y_m, y_0\rangle]$ with a palindrome string $\alpha \in \mathcal{L}^{2n-1}$, i.e, $\Lambda_{y_2}([\langle y_2, y_3\rangle, \ldots, \langle y_m, y_0\rangle]) = \Lambda_{y_0}([\langle y_0, y_m\rangle, \langle y_m, y_{m-1}\rangle, \ldots, \langle y_3, y_2\rangle]) = \alpha$ (this is possible, since the length of the path is odd). This labeling of $\widehat{R_{2n+1}}(1,1)$ is clearly a $\mathcal{LSO}$. Complete the labeling of G maintaining the property of $\mathcal{LSO}$ (this is possible by Lemma 3).
Consider the path $\pi_1 = [u, y_0, y_m, y_{m-1}, \ldots y_2, v] \in P[u, v]$ and the cycle $\pi_2 = [y_1, \ldots, y_m, y_0, y_1] \in P[y_1, y_1]$; we have that $\Lambda_x(\pi_1) = [a \circ \alpha \circ b] = \Lambda_{y_0}(\pi_2)$.
By Lemma 2 it follows that λ is inconsistent, yielding the contradiction.

Case d): G contains $R_{4;1}$.
In $R_{4;1}$, let $[x, y, z, w]$ be the nodes of the ring, and let $\langle y, w\rangle$ be the chord. Let $a, b, c, \bar{a}, \bar{b}$ be five distinct labels. Label the edges $\langle x, y\rangle$ and $\langle w, z\rangle$ with a; label $\langle y, x\rangle$ and $\langle z, w\rangle$ with $\bar{a}$; label $\langle y, w\rangle$ and $\langle z, y\rangle$ with b; label $\langle w, y\rangle$ and $\langle y, z\rangle$ with $\bar{b}$; and label $\langle x, w\rangle$ and $\langle w, x\rangle$ with c. This labeling of $R_{4;1}$ is clearly a $\mathcal{LSO}$.
Consider the cycle $\pi_1 = [w, y, x, w] \in P[w, w]$ and the path $\pi_2 = [y, z, w, x] \in P[y, x]$; we have that $\Lambda_w(\pi_1) = [\bar{b}, \bar{a}, c] = \Lambda_y(\pi_2)$.
By Lemma 2 it follows that λ is inconsistent, yielding the contradiction. □
In other words, given a labeled graph (G, λ), for λ to be globally consistent consistent, and, thus, to be a $\mathcal{SD}$, it is necessary that G does not contain any of the graphs indicated above.

In the following, the $(1, 0)$-hat ring of even size, the $(2, 0)$-hat ring, $(1, 1)$-hat ring of odd size, and the ring of size four with one chord $R_{4;1}$ will be called *LSO-forbidden graphs* (see Figure 4).

5. Feasible graphs for $\mathcal{LSO}$

In this Section, we characterize the class of graphs (*LSO-feasible graphs*) in which every locally symmetric orientation is globally consistent.

Given a graph (G, λ), let $\mathcal{H}_{G,\lambda} = \{\alpha \in \mathcal{L}^+ : \exists x, y \in V, \pi \in P[x, y],\ \alpha = \Lambda_x(\pi)\}$ be the set of the sequences of labels which correspond to actual walks in the graph (in the following, when no ambiguity arises, the subscript G, λ will be omitted).

5.1. Trees

Definition 9 Simple contraction
Given $\alpha \in \mathcal{L}^+$, a simple contraction $\overline{\alpha}$ *of α is the sequence of labels obtained from α by repeatedly deleting every pair of consecutive labels l, l' such that $l = \psi(l')$.*

Theorem 5 *Locally symmetric orientation is sufficient for $\mathcal{SD}$ in trees.*

Proof. Let T_n be a tree of size n whose labeling λ is a $\mathcal{LSO}$. Let ψ be the edge symmetry function. Consider the function β defined as follows: $\forall x, y \in V$, $\beta_x(y)$ is the sequence of labels corresponding to the shortest path between x and y. Since the graph is a tree, $\beta_x(y) = \beta_x(z)$ iff $y = z$; thus, β is a naming function.
Consider the coding function f defined as follows: $\forall \pi \in P[x]$, $f(\Lambda_x(\pi)) = \overline{\Lambda_x(\pi)}$. Clearly, $\forall \pi \in P[u, v]$, $f(\Lambda_u(\pi)) = \beta_u(v)$; thus, f is consistent.
It is easy to see that the following decoding function is consistent as well: $\forall \langle x_0, y_0\rangle \in E(x_0)$, $\forall \pi \in P[y_0]$, $\pi = [\langle y_0, y_1\rangle, \ldots, \langle y_{m-1}, y_m\rangle]$: $h(\lambda_{x_0}(\langle x_0, y_0\rangle), f(\Lambda_{y_0}(\pi))) = f(\lambda_{x_0}(\langle x_0, y_0\rangle) \circ f(\Lambda_{y_0}(\pi))$. □

5.2. Rings

Let (R, λ) be a labeled ring of size n with $\mathcal{LSO}$; let the nodes be $[x_1, \ldots, x_n]$ and ψ be the edge symmetry function. Let R_i^+ (R_i^-) be the sequence of labels on the ring starting from x_i in clockwise (counterclockwise) direction, and let $\mathcal{R} = \{R_i^+ \cup R_i^- : 1 \le i \le n\}$.

Lemma 4 *Let* $\alpha \in \mathcal{H}$. *If* $\alpha \in \mathcal{R}$, *there do not exist* $y, z \in V$ *and* $\pi \in P[y,z]$, *such that* $y \neq z$ *and* $\alpha = \Lambda_y(\pi)$.

Proof. By contradiction, suppose that there exist $y \neq z$ and $\pi \in P[y,z]$, such that $\alpha = \Lambda_y(\pi)$. Since $\alpha \in \mathcal{R}$, $|\alpha| = |\pi| = n$. Since $|\pi| = n$ and $y \neq z$, the walk must contain at least a cycle; let $\pi' = [x_i,\ x_{i+1},\ \ldots,\ x_j,\ x_{j-1},\ \ldots,\ x_i]$ be such a cycle in π. Let $\lambda_{x_{j-1}}(\langle x_{j-1}, x_j\rangle) = a$; since the system has edge symmetry, we have that $\lambda_{x_j}(\langle x_j, x_{j-1}\rangle) = \psi(a)$. Since $\alpha \in \mathcal{R}$, then, the labels a and $\psi(a)$ must appear in R consecutively in the same direction (say, clockwise): let $\lambda_{x_{h-1}}(\langle x_{h-1}, x_h\rangle) = a$ and $\lambda_{x_h}(\langle x_h, x_{h+1}\rangle) = \psi(a)$. Thus, we have that $\lambda_{x_{h+1}}(\langle x_{h+1}, x_h\rangle) = a$ violating the local orientation in x_h. □

Definition 10 Circular contraction
Given $\alpha \in \mathcal{H}$, *a* circular contraction *of* α *with respect to* $\mathcal{R}$ *is the sequence* $\tilde{\alpha}_{\mathcal{R}}$ *obtained from* α *by:*
1) deleting every pair of consecutive labels l, l' *such that* $l = \psi(l')$
2) deleting every sequence of n *consecutive labels* $[l_1, \ldots, l_n] \in \mathcal{R}$
3) repeating operations 1) and 2) until they are no longer applicable.

Lemma 5 *Let* $\alpha \in \mathcal{H}$ *be a walk in* $P[x,y]$, *then:*
i) if $x = y$, $\tilde{\alpha}_{\mathcal{R}}$ *is the empty string;*
ii) if $x \neq y$, $\tilde{\alpha}_{\mathcal{R}} = \Lambda_x(\pi')$, *where* π' *is a path in* $P[x,y]$.

Proof. It directly follows from the definition of circular contraction and from Lemma 4. □

Consider the following relation $\rightarrow_{\mathcal{R}}$ on sequences of labels $\mathcal{H}$ with respect to $\mathcal{R}$.

Definition 11 *Given* α, $\gamma \in \mathcal{H}$, *let* $\tilde{\alpha}_{\mathcal{R}} = [a_1, \ldots, a_h]$ *and* $\tilde{\gamma}_{\mathcal{R}} = [b_1, \ldots b_k]$. *Then,* $\alpha \rightarrow_{\mathcal{R}} \gamma$ *iff:*
i) $\tilde{\alpha}_{\mathcal{R}}$ *and* $\tilde{\gamma}_{\mathcal{R}}$ *are both the empty string; or*
ii) $[a_1, \ldots, a_h, \psi(b_k), \ldots, \psi(b_1)]$ *is the sequence of labels corresponding to a cycle.*

From Lemma 5, it follows that:

Lemma 6 *Given* α, $\gamma \in \mathcal{H}$, *let* $\tilde{\alpha}_{\mathcal{R}} = [a_1, \ldots, a_h]$, $\tilde{\gamma}_{\mathcal{R}} = [b_1, \ldots b_k]$. *Then* $\alpha \rightarrow_R \gamma$ *iff one of the following conditions holds:*
i) $\tilde{\alpha}_{\mathcal{R}}$ *and* $\tilde{\gamma}_{\mathcal{R}}$ *are both the empty string;*
ii) $\alpha = \gamma$;
iii) $[a_1,\ \ldots,\ a_h,\ \psi(b_k),\ \ldots,\ \psi(b_1)] \in \mathcal{R}$.

Let $\star_R$ be the transitive closure of the relation $\rightarrow_R$. We have that:

Lemma 7 *Relation* $\star_R$ *is an equivalence relation on* $\mathcal{H}$.

Proof. Given $\alpha, \gamma \in \mathcal{H}$, let $\tilde{\alpha}_{\mathcal{R}} = [a_1, \ldots, a_h]$, $\tilde{\gamma}_{\mathcal{R}} = [b_1, \ldots b_k]$.
$\star_{\mathcal{R}}$ is reflexive, in fact $\alpha \star_{\mathcal{R}} \alpha$ by Lemma 6.
Let $\alpha \star_{\mathcal{R}} \gamma$. If $\tilde{\alpha}_{\mathcal{R}}$ and $\tilde{\gamma}_{\mathcal{R}}$ are both the empty strings, or if $\alpha = \gamma$, we clearly have that $\gamma \star_{\mathcal{R}} \alpha$; otherwise, by Lemma 6 it follows that $[a_1,\ \ldots,\ a_h,\ \psi(b_k),\ \ldots,\ \psi(b_1)] \in \mathcal{R}$. In this case, we have that also $[b_1,\ \ldots,\ b_k,\ \psi(a_h),\ \ldots,\ \psi(a_1)] \in \mathcal{R}$, which means that $\gamma \star_{\mathcal{R}} \alpha$. Thus, $\star_{\mathcal{R}}$ is symmetric.
Furthermore, $\star_{\mathcal{R}}$ is transitive by definition. □

Let $[\alpha]_{\star_R}$ denote the equivalence class containing α.

Theorem 6 *Locally symmetric orientation is sufficient for $\mathcal{SD}$ in rings.*

Proof. Let R be a ring whose labeling λ is a $\mathcal{LSO}$. Let ψ be the edge symmetry function.

Consider the function β defined as follows: $\forall x, y \in V$, if $x \neq y$, $\beta_x(y) = [\alpha]_{\star_R}$, where $\alpha = \Lambda_x(\pi)$, and $\pi \in P[x, y]$ is the path from x to y following the ring in clockwise-order; otherwise, $\beta_x(y)$ is the empty string.

Since $\star_{\mathcal{R}}$ is an equivalence relation, it follows that $\beta_x(y) = \beta_x(z)$ iff $y = z$; thus, β is clearly a naming function.

Consider the coding function f defined as follows: $\forall \pi \in P[x, y]$, $\alpha = \Lambda_x(\pi)$ $f(\alpha) = [\tilde{\alpha}]_{\star_R}$. By Lemma 5, if $x \neq y$, $\tilde{\alpha}$ corresponds to a path from x to y; otherwise it is the empty string. Let γ be the path in clockwise direction from x to y. By definition of naming function, $[\gamma]_{\star_{\mathcal{R}}} = \beta_x(y)$. Let $\gamma = [b_1, \ldots, b_h]$; since $[\tilde{\alpha}, \psi(b_h), \ldots \psi(b_1)]$ corresponds to a cycle or is the empty string, and since $\alpha \in [\gamma]_{\star_R}$, it follows that $[\tilde{\alpha}]_{\star_R} = [\gamma]_{\star_R}$, thus, $f(\alpha) = \beta_x(y)$ and the coding function is consistent.

It is easy to see that the following decoding function is consistent as well: $\forall \langle x, y\rangle \in E(x)$, $\forall \pi \in P[y]$: $h(\lambda_x(\langle x, y\rangle), f(\Lambda_y(\pi))) = f(\lambda_x(\langle x, y\rangle) \circ \gamma)$, where γ is an arbitrary element of the equivalence class $f(\Lambda_y(\pi))$. □

5.3. Hat-free spiders

Given a hat-free spider of odd size $\overline{S}_{2n+1}$, let $m = 2n$, $R = [x_0, \ldots, x_m]$ be the nodes of the ring. Let R_i^+ (R_i^-) denote the sequence of labels on the ring starting from x_i in clockwise (counterclockwise) direction. Let $\mathcal{R} = \{R_i^+ \cup R_i^- \ : \ 1 \leq i \leq n\}$.

An obvious property of hat-free spiders is the following:

Lemma 8 *Any path of length m contains one edge of a spike.*

Lemma 9 *Let $\alpha \in \mathcal{H}$. If α corresponds to a cycle, there do not exist any $y, z \in V$ and any $\pi \in P[y, z]$ such that $y \neq z$ and $\alpha = \Lambda_y(\pi)$.*

Proof. Let $\alpha \in \mathcal{R}$. We will now show that there are no paths with label α.

By contradiction, let π be a path with labels α. By Lemma 8, π contains one edge of a spike; without loss of generality, let $\langle x_0, y_0\rangle$ be such an edge. The only possible situations in which π is a path are when: *a)* $\pi \in P[y_0, x_1]$, or *b)* $\pi \in P[y_0, x_m]$, or *c)* $\pi \in P[x_m, y_0]$, or *d)* $\pi \in P[x_1, y_0]$.

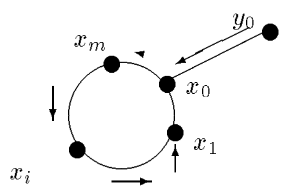


Figure 5: Proof of Lemma 8.

Consider situation *a)*. Let $\pi \in P[y_0, x_1]$, $\pi = [\ \langle y_0, x_0\rangle, \langle x_0, x_m\rangle, \ldots, \langle x_2, x_1\rangle]$ (see Figure 5). By contradiction, let $\alpha = \Lambda_{y_0}(\pi)$.

Since, by definition, $\alpha \in \mathcal{R}$ and since there is edge symmetry, there must exist a node x_i such that $\Lambda_{x_i}([\langle x_i, x_{i+1}\rangle, \langle x_{i+1}, x_{i+2}\rangle, \ldots, \langle x_{i-1}, x_i\rangle]) = \Lambda_{y_0}([\langle y_0, x_0\rangle, \langle x_0, x_m\rangle, \ldots, \langle x_2, x_1\rangle])$. In particular, we must have that: $\Lambda_{x_{i+1}}([\langle x_{i+1}, x_{i+2}\rangle, \ldots, \langle x_m, x_0\rangle]) = \Lambda_{x_0}([\ \langle x_0, x_m\rangle, \ldots, \langle x_{i+2}, x_{i+1}\rangle])$. This implies that $[\langle x_{i+1}, x_{i+2}\rangle, \ldots, \langle x_m, x_0\rangle]$, must be labeled with a palindrome string; thus, it must be of odd length, i.e. $m - (i+1) + 1$ is odd, and, thus $m - i$ is odd.
On the other hand, by the same reasoning, $\Lambda_{x_{i+1}}([\langle x_{i+1}, x_i\rangle, \langle x_i, x_{i-1}\rangle, \ldots, \langle x_2, x_1\rangle]) = \Lambda_{x_0}([\langle x_0, x_1\rangle, \ldots, \langle x_{i-1}, x_i\rangle])$, which implies that also $[\langle x_1, x_2\rangle, \ldots, \langle x_{i-1}, x_i\rangle]$ must be labeled with a palindrome string; thus, it must be of odd length, i.e. $i - 1$ must be odd.
Since both $m - 1$ and $i - 1$ are odd, it follows that m is odd and, thus, the ring has even size. This contradicts our hypothesis.

The cases b), c), and d) can be proved using a similar reasoning. □

Lemma 10 *Let $\alpha \in \mathcal{H}$ be a walk in $P[x, y]$, then:*
i) if α corresponds to a cycle, $\tilde{\alpha}_{\mathcal{R}}$ is the empty string;
ii) otherwise, $\tilde{\alpha}_{\mathcal{R}} = \Lambda_x(\pi')$, where π' is a path in $P[x, y]$.

Proof. It directly follows from the definition of circular contraction and from Lemma 9. □

From Lemma 10, it follows that:

Lemma 11 *Given α, $\gamma \in \mathcal{H}$, let $\tilde{\alpha}_{\mathcal{R}} = [a_1, \ldots, a_h]$, and $\tilde{\gamma}_{\mathcal{R}} = [b_1, \ldots b_k]$. Then, $\alpha \rightarrow_{\mathcal{R}} \gamma$ iff one of the following conditions holds:*
i) $\tilde{\alpha}_{\mathcal{R}}$ and $\tilde{\gamma}_{\mathcal{R}}$ are both the empty string;
ii) $\alpha = \gamma$;
iii) $[a_1, \ldots, a_h, \psi(b_k), \ldots, \psi(b_1)] \in \mathcal{R}$;
iv) $a_1 = b_1$ and $[a_2, \ldots, a_h, \psi(b_k), \ldots, \psi(b_2)] \in \mathcal{R}$;
iv) $a_h = b_k$ and $[a_1, \ldots, a_{h-1}, \psi(b_{k-1}), \ldots, \psi(b_2)] \in \mathcal{R}$.

Let $\star_{\mathcal{R}}$ be the transitive closure of the relation $\rightarrow_R$. It is easy to see that:

Lemma 12 *Relation $\star_{\mathcal{R}}$ is an equivalence relation on $\mathcal{H}$.*

Proof. From Lemma 11 it directly follows that $\star_{\mathcal{R}}$ is reflexive and symmetric. Furthermore, $\star_{\mathcal{R}}$ is transitive by definition. □

Let $[\alpha]_{\star_{\mathcal{R}}}$ denote the equivalence class containing α.

Theorem 7 *Locally symmetric orientation is sufficient for $\mathcal{SD}$ in hat-free spiders of odd size $\overline{S}_{2n+1}$.*

Proof. Let $\overline{S}_{2n+1}$ be a hat-free spider of odd size whose labeling λ is a $\mathcal{LSO}$ and let $m = 2n$. Let ψ be the edge symmetry function, and let $[x_0, x_1 \ldots x_m]$ be the nodes of the ring.

Consider the function β defined as follows: $\forall x, y \in V$, if $x \neq y$, $\beta_x(y) = [\alpha_{xy}]_{\star_{\mathcal{R}}}$ where α_{xy} is the sequence of labels corresponding to the path from x to y following the ring in clockwise-order; otherwise, $\beta_x(y)$ is the empty string.
Function β is a naming function, in fact we will show that $\beta_x(y) = \beta_x(z)$ iff $y = z$.
If $y = z$, then $\beta_x(y) = \beta_x(z)$, $\forall x$.

Let $y \neq z$; we now show that $\beta_x(y) \neq \beta_x(z)$. By contradiction, suppose $\beta_x(y) = \beta_x(z)$. If $x = y$, since $\beta_x(x)$ is the empty string by definition, $x = y$ implies $y = z$ (and viceversa). Let both y and z be distinct from x; then $\beta_x(y) = \beta_x(z)$ implies $[\alpha_{xy}]_{\star\mathcal{R}} = [\alpha_{xz}]_{\star R}$. By Lemma 11, it follows that the labels $[\Psi(\alpha_{x,y}) \circ \alpha_{x,z}]$ correspond to a cycle, and hence $y = z$.

Consider the coding function f defined as follows: $\forall \pi \in P[x, y]$, $\alpha = \Lambda_x(\pi)$ $f(\alpha) = [\tilde{\alpha}]_{\star\mathcal{R}}$ if $x \neq y$, otherwise, $f(\alpha)$ is the empty string. To prove that it is consistent we must show that $f(\alpha) = \beta_x(y)$. By Lemma 10, $\alpha_{x,y}$ is the sequence of labels corresponding to a path from x to y. Let $\beta_x(y) = [\gamma]_{\star\mathcal{R}}$. By definition of naming function $[\alpha \circ\ \Psi(\gamma)]$ corresponds to one of the condition of Lemma 11; that is, $\alpha \in [\gamma]_{\star\mathcal{R}}$, and, thus, $f(\alpha) = \beta_x(y)$ and the coding function is consistent.

It is easy to see that the following decoding function is consistent as well: $\forall \langle x, y\rangle \in E(x)$, $\forall \pi \in P[y]$: $h(\lambda_x(\langle x, y\rangle), f(\Lambda_y(\pi))) = f(\lambda_x(\langle x, y\rangle) \circ \gamma$, where γ is an arbitrary element of the equivalence class $f(\Lambda_y(\pi))$. □

An example of hat-free spider of odd size is shown in Figure 6.

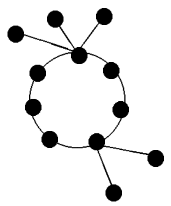


Figure 6: An hat-free spider.

6. Necessary and Sufficient Condition for $\mathcal{LSO}$

In this section, we show that the characterization is complete, in fact a graph either is feasible, or it contains a forbidden subgraph. We can thus conclude that every $\mathcal{LSO}$ is globally consistent in a graph iff the graph does not contain any forbidden subgraph.

Theorem 8 *Given a graph G, either G is LSO-feasible, or it contains a LSO-forbidden graph.*

Proof. If G is a tree, by Theorem 5, the theorem holds. Let then G contain a cycle. If G is a ring, by Theorem 6, the theorem holds. To complete the proof we must show that the theorem holds when G contains a cycle of length smaller than the number of nodes.

We shall consider two cases depending on whether the length of such a cycle is even or odd.

Case (1). Even length cycle.

Let $[x_0, \ldots x_{2n-1}]$ be the nodes of such a cycle. Since G is not a ring, the cycle has (1.1) one or more spikes, or (1.2) one or more internal chords.

Case (1.1). If there exists a spike of any length, G would contain a hat ring $\widehat{R_{2n}}(1,0)$, which is LSO-forbidden by Theorem 4.

Case (1.2). Let $\langle x_i, x_j \rangle$ be a chord. The chord divides the cycle in two parts $[x_i, x_{i+1}, \ldots, x_{j-1}, x_j]$ and $[x_j, x_{j+1}, \ldots, x_{i-1}, x_i]$ (here and in the following, all the operations are modulo $2n$). We must have that either $(j-i)$ and $(i-j)$ are both even or they are both odd. Suppose they are both even: $(j-i) = 2k$ and $(i-j) = 2h$ (with $k \leq h$). In this case, if $n = 2$, G would contain $R_{4;1}$; if $n > 2$ it would contain a spiked ring $\widehat{R_{2k+1}}(2,0)$ (see Figure 7). In both cases G would contain a LSO-forbidden subgraph (Theorem 4).
If they are both odd, let $(j-i) = 2k+1$ and $(i-j) = 2l+1$ (with $k \leq l$); the graph G would contain a spiked ring $\widehat{R_{2k+2}}(1,0)$ which, by Theorem 4, is LSO-forbidden.

$n = 2$ $\qquad$ $n > 2$

Figure 7: Proof of Case 1.2.

Case (2). Odd length cycle.

Let $[x_0, \ldots x_{2n}]$ be the nodes of such a cycle. Since G is not a ring, the cycle has (2.1) a chord, or (2.2) a spike of length greater than 1, or (2.3) only spikes of length 1.

Case (2.1). The chord divides the cycle in two parts $[x_i, x_{i+1}, \ldots, x_{j-1}, x_j]$ and $[x_j, x_{j+1}, \ldots, x_{i-1}, x_i]$ (here and in the following, all the operations are modulo $2n+1$), one of which has odd length, while the other has even length. Let $(j-i) = 2k$, and $(i-j) = 2h+1$. In this case G would contain a hat ring $\widehat{R_{2h+2}}(1,0)$ which, by Theorem 4, is LSO-forbidden.

Case (2.2). If the cycle has a spike of length more than one, G would contain a hat ring $\widehat{R_{2n+1}}(2,0)$ which, by Theorem 4, is LSO-forbidden.

Case (2.3). The only case left to consider is when in G all cycles are of odd length, contain no chords, and have no spikes of length greater than 1. First of all observe that, to satisfy such requirements, G must have only one cycle (otherwise the edges of the other cycle would form a spike of length greater than 1 for the first cycle). In other words, G is a spider. Then, it either contains $\widehat{R_{2n+1}}(1,1)$, which is LSO-forbidden by Theorem 4, or it is a hat-free spider which, by Theorem 7, is LSO-feasible.

□

From Theorems 4, 5, 6, 7 and from Theorem 8, it follows that

Theorem 9 *Every Locally symmetric orientation λ is globally consistent in G iff G does not contain any LSO-forbidden graph.*

7. Conclusions

In this paper we have investigated the strong interplay between the structure of a distributed system and the consistency properties which a labeling must satisfy to be a Sense of Direction.

In particular, we have studied what topological constraints the labelings satisfying a given set of properties $\mathcal{P}$ will always be globally consistent.

We have provided a complete answer to the above questions for two large classes of labelings, *Local Orientations* and (the commonly used) *Locally Symmetric Orientations*.

For each class, we have identified and characterized both the set of *forbidden subgraphs* and the set of *feasible graphs*. We have shown that this characterization is complete: in fact, every graph is either feasible or contains a forbidden subgraph.

Not surprisingly, we have shown that local orientation suffices for Sense of Direction only in trivial graphs. On the other hand, we have proved that locally symmetric orientations, guarantees Sense of Direction in a large class of graphs.

A major contribution of this paper has been to move the first step toward answering the more general question:
What is the nature of the relationship between network topology and Sense of Direction?.

Many intriguing related questions are still open, all of them of immediate practical relevance; for example:
What is the impact of network topology on other classes of labelings?
Which topological properties guarantee the existence of a "minimal" Sense of Direction (i.e., with smallest number of labels)?
What is the complexity of deciding if a labeling is a Sense of Direction?

Investigations in these directions have already started, providing the first partial results [43,44].

Acknowledgments

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