Minimal Sense of Direction and Decision Problems for Cayley Graphs

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Abstract

Sense of direction is a property of the labelling of (possibly anonymous) networks which allows to assign coherently local identifiers to other processors on the basis of the route followed by incoming messages. A graph has *minimal sense of direction* whenever it has sense of direction and the number of colours equals its maximum outdegree. We prove that an outregular digraph with minimal *weak* sense of direction is a Cayley colour graph (in the general sense, i.e., we do not require connectedness). Since Cayley colour graphs are known to possess minimal *transitive* sense of direction, we obtain a characterization of outregular graphs with minimal (weak,transitive) sense of direction. As a consequence, deciding whether a coloured graph is a Cayley colour graph reduces to deciding whether it has weak sense of direction, which can be done in AC^1.

Keywords: distributed systems; distributed computing; computational complexity; sense of direction; Cayley graphs

1 Introduction

The topological structure of distributed systems can be described by graphs, with nodes representing agents and arcs representing links. Each node has a local (partial) view of the system, and it associates a different label (colour) to each of its incident links.

The solution to many problems in a distributed system can be greatly simplified by using colourings with special properties. In particular, in this paper we study *sense of direction*, a property of global consistency [San84, FMS95]. In a previous paper [BV96], the authors characterized weak sense of direction in a combinatorial manner, and showed that it can be decided efficiently in parallel; moreover, they analogously showed that sense of direction is decidable in sequential polynomial time.

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A challenging problem is that of finding a colouring which gives (weak) sense of direction to a given graph using a small number of colours. The problem of finding a *minimal sense of direction* [Flo97] consists in determining a colouring of the arcs which uses a number of colours equal to the maximum degree (it is impossible to find a sense of direction with fewer colours, because the graph must be deterministic).

In this paper we show that if a directed graph with regular outdegree has weak sense of direction then it is a Cayley colour graph. Since all such graphs possess a minimal transitive sense of direction, we obtain an exact characterization of minimality for outregular graphs. A special case of this result, for symmetric connected graphs, has been obtained independently in [FRS96], and it is equivalent to the known characterization of undirected Cayley colour graphs given in [MKS76].

A consequence of our result is that one can use the algorithms proposed in [BV96] in order to recognize a Cayley colour graph in AC^1. This of course opens the problem of recognizing a Cayley graph without knowing its natural colouring, and we conclude by conjecturing that this is at least as hard as graph isomorphism.

2 Definitions

A *directed graph* (or, in short, a graph) G is given by a set V of vertices and a set $A \subseteq V \times V$ of arcs. We write $P[x, y] \subseteq A^*$ for the set of paths from the vertex x to the vertex y.

Let Γ be a finite group, and $S \subseteq \Gamma$ a set of elements of Γ (which are usually taken to be generators). The *Cayley graph* of Γ with respect to S is the graph having vertex set Γ, and an arc from g to h whenever there is an $s \in S$ such that $gs = h$. When S is closed under inversion, the resulting graph is symmetric (or, equivalently, it is undirected). Note that the graph is $|S|$-regular (i.e., there are exactly $|S|$ arcs going into and $|S|$ arcs going out of every vertex), and that it is strongly connected iff S is a set of generators.

An *(arc) colouring* of a graph G is a function $\lambda : A \to \mathscr{L}$, where $\mathscr{L}$ is a finite set of colours. We say that λ is *deterministic* iff

$$\lambda(\langle x, y\rangle) = \lambda(\langle x, z\rangle) \implies y = z$$

i.e., if the automaton described by the transition graph G with colouring λ is deterministic. Note that for each Cayley graph there exists a natural deterministic colouring in the set S: an arc from g to gs is coloured by s. Such a graph is called the *Cayley colour graph* of Γ with respect to S.

Given a graph G coloured by λ, let

$$L(x, y) = \{\lambda^*(\pi) \mid \pi \in P[x, y]\},$$

where $\lambda^* : A^* \to \mathscr{L}^*$ is the natural extension of λ. In other words, $L(x, y)$ is the *language* recognized by G when x is the initial state and y is the final state. For all $I \subseteq V^2$ let

$$L_I = \bigcup_{\langle x,y\rangle \in I} L(x, y)$$

(of course, $L(x, y) = L_{\{\langle x,y\rangle\}}$). Notice that $\varepsilon \in L(x, x) \neq \varnothing$.

A *local naming* for G is a family of injective functions $\beta = \{\beta_x : V \to \mathscr{N}\}_{x \in V}$, with $\mathscr{N}$ a finite set, called the *name space*. Intuitively, each vertex x of G gives to each other vertex y a name $\beta_x(y)$ taken from the name space.

Given a coloured graph endowed with a local naming, a function $f : L_{V^2} \to \mathscr{N}$ is a *coding function* iff

$$\forall x, y \in V \quad \forall \pi \in P[x, y] \qquad f(\lambda^*(\pi)) = \beta_x(y).$$

A coding function translates the colouring of the path along which two vertices x, y are connected into the name which x gives to y. Note that while the resulting name is *local* (i.e., x and z might choose different elements of the name space for the same vertex y), the coding function is *global* (i.e., it is the same for all vertices).

A colouring λ is a *weak sense of direction* for a graph G iff for some local naming there is a coding function. We shall also say that a coloured graph *has* weak sense of direction, or that λ *gives* weak sense of direction to G. (We remark that some authors consider also the *nonhomonymous* case, in which only nonempty paths are taken into account; in this setting, $\beta_x(x)$ is not independent of x.)

Finally, a colouring λ is a *transitive* sense of direction if there is a local naming and a coding function such that the name space $\mathscr{N}$ is a multiplicative group satisfying the condition

$$\beta_x(y)\beta_y(z) = \beta_x(z)$$

for all vertices x, y and z. It has been proved in [BV97] that there are graphs with a weak sense of direction which is not transitive.

3 Minimal sense of direction

The number of colours which is necessary in order to give to a graph G with n vertices a sense of direction is always bounded from below by the maximum outdegree of the graph, and from above by the number of vertices (order arbitrarily the vertices and then assign to each arc from the i-th to the j-th vertex the colour $(i - j) \bmod n$, n being the number of vertices). When a graph G possesses a colouring λ achieving the lower bound, we say that λ is a *minimal* sense of direction: an example is given by Cayley colour graphs. In the case of a complete graph (with self-loops) the upper and lower bounds coincide.

The following definition will be intensively used throughout the paper. Note that we do not make distinctions between partitions and equivalence relations.

Definition 1 Given a set X, a partition (or equivalence relation) Π of X^2 is said to be *(internally) monodrome* iff its elements are graphs of partial functions from X to X, i.e., iff

$$\forall I \in \Pi \qquad \langle x, y\rangle, \langle x, z\rangle \in I \implies y = z.$$

It is known from [BV96] that the colouring of a graph induces on the set V^2 a certain equivalence relation, whose monodromy is equivalent to having weak sense of direction; more precisely, we have the following

Theorem 1 Let G be a graph coloured by λ, and T the transitive closure of the relation $\sim$ on V^2 defined by

$$\langle x, y\rangle \sim \langle x', y'\rangle \iff L(x, y) \cap L(x', y') \neq \varnothing.$$

Then λ is a weak sense of direction iff T is monodrome.

Recall that a graph is *d-outregular* iff there are exactly d arcs outgoing from each vertex. The main result we shall prove is the following one:

Theorem 2 Let G be a d-outregular coloured graph which has minimal weak sense of direction. Then G is a Cayley colour graph induced by a subset of d elements.

As immediate corollaries, we obtain:

Corollary 1 An outregular coloured graph has minimal (weak,transitive) sense of direction if and only if it is a Cayley colour graph.

Corollary 2 An outregular graph can be given a minimal (weak,transitive) sense of direction if and only if it is a Cayley graph.

Note that since Cayley graphs possess a transitive sense of direction, a d-outregular graph with a weak sense of direction has a transitive sense of direction (in fact, with the same colouring); moreover, the graph is also d-inregular.

From now onwards, let G be a d-outregular graph (with vertex set V), and λ be a colouring which is a *minimal* weak sense of direction for G (and thus $|\mathscr{L}| = d$). We note that the presence of multiple parallel arcs or self-loops would not change the essence of our result. In fact, in a graph with a given weak sense of direction the colours associated to loops are disjoint from the colour associated to non-loops, and similarly if two parallel arcs coloured by α and β exist than every pair of arcs outgoing from a vertex and coloured by α and β must be parallel. On the other hand, if we are searching for a good sense of direction, we can always assume to colour parallel arcs in the same way, and choose a fixed colour for loops.

First of all observe that, for all $w \in \mathscr{L}^*$ and $x \in V$, there is a unique vertex $x \cdot w \in V$ such that $w \in L(x, x \cdot w)$ (in automata-theoretic terms, it is the state reached from x on input w by the automaton having G as transition graph).

Proposition 1 With G, λ as above:

1. for all $\alpha \in \mathscr{L}$ there exists $k_\alpha > 0$ such that
$$\forall x \in V \quad x \cdot \alpha^i = x \text{ iff } k_\alpha \text{ divides } i;$$

2. if there is a path in G from x to y, then there is also a path from y to x;

3. every weakly connected component of G is strongly connected.

Proof. (1) Take any $x \in V$ and consider the sequence $x = x \cdot \varepsilon, x \cdot \alpha, x \cdot \alpha^2, \ldots$. Clearly, at some point we shall visit a vertex which has already been visited, i.e., there are $i < j$ such that $x \cdot \alpha^i = x \cdot \alpha^j$ (we assume that j is the first such index). Let $k_\alpha = j - i$; of course $i = 0$, for otherwise α^{k_α} would colour both a cycle (from $x \cdot \alpha^i$ to $x \cdot \alpha^j$) and a non-cycle (from $x \cdot \alpha^{i-1}$ to $x \cdot \alpha^{j-1}$), which contradicts the definition of weak sense of direction. Then, the result follows immediately.
(2) Suppose $x \cdot w = y$, and let $w = \alpha_1 \cdots \alpha_p$. Then let $v = \alpha_p^{k_{\alpha_p}-1} \cdots \alpha_2^{k_{\alpha_2}-1} \alpha_1^{k_{\alpha_1}-1}$; it is easy to see that $y \cdot v = x$.
(3) By using (2), we can turn each walk of G into a path in any of the two possible directions. ∎

Now we proceed with the proof of our main result. Consider a class $I \in T$ (the definition of T has been given in the statement of Theorem 1); there are two cases:

- $L_I = \varnothing$ and I is a singleton (corresponding to a pair of vertices lying in two different components of G);
- $L_I \neq \varnothing$ and I is the graph of a total function. In fact, let $w \in L_I$ and $x \in V$: clearly $\langle x, x \cdot w \rangle \in I$. So, for all $x \in V$ there is a $y \in V$ such that $\langle x, y \rangle \in I$ (y is unique by monodromy).

Note that in the second case for all $\langle x, y \rangle, \langle x', y' \rangle \in I$ we have $L(x, y) = L(x', y') = L_I$. In fact, let $w \in L(x, y)$. Of course, $\langle x', x' \cdot w \rangle \in I$ and so $x' \cdot w = y'$. Thus $w \in L(x', y')$.

Let now $R = \{I \in T : L_I \neq \varnothing\}$; each I in this set is the graph of a total function from V to V (by the second of the above remarks).

Moreover, suppose that $\langle x, z \rangle, \langle y, z \rangle \in I \in R$; if $w = \alpha_1 \alpha_2 \cdots \alpha_p \in L(x, z) = L(y, z)$, then $\alpha_p^{k_{\alpha_p}-1} \cdots \alpha_1^{k_{\alpha_1}-1}$ should colour a path from z to x *and* from z to y, which is impossible, unless $x = y$. Thus, each $I \in R$ is simply a permutation of V.

Finally, observe that, for all $I, J \in R$:

- if $\langle x, y\rangle, \langle x', y'\rangle \in I$ and $\langle y, z\rangle, \langle y', z'\rangle \in J$ then $\langle x, z\rangle$ and $\langle x', z'\rangle$ belong to the same element of R; in other words, R (seen as set of permutations) is closed under composition;

- if $\langle x, y\rangle \in I$, then $\langle y, x\rangle$ belongs to some class of R (a consequence of Proposition 1(2)). So R is also closed under inversion;

- R contains the class $1 = \{\langle x, x\rangle : x \in V\}$, corresponding to the identical permutation.

To summarize, R is a group. For each $\alpha \in \mathscr{L}$, let I_α be the element of R such that $\alpha \in L_{I_\alpha}$. Clearly, each class of R can be obtained by composing the I_α's, so $S = \{I_\alpha\}_{\alpha\in\mathscr{L}}$ is a set of generators for R (note that $I_\alpha \neq I_\beta$ for distinct α and β).

Let now H be the Cayley graph of R w.r.t. S, with its natural colouring. Let $U \subseteq V$ be any connected component of G, and $\overline{x} \in U$. Define:

$$\begin{array}{rccc} \varphi : & U & \to & R \\ & x & \mapsto & [\langle \overline{x}, x\rangle] \end{array}$$

where $[\langle x, y\rangle]$ is the class of R including the pair $\langle x, y\rangle$. Clearly, φ is well-defined, because U is (strongly) connected (see Proposition 1(3)). Moreover:

- φ is injective (by monodromy), and also onto (if $I \in R$, just let $w \in L_I$; we have $\varphi(\overline{x} \cdot w) = [\langle \overline{x}, \overline{x} \cdot w\rangle] = I$);

- suppose there is an arc of G from x to y coloured by α, and let $w \in L(\overline{x}, x)$. Of course, $w\alpha \in L(\overline{x}, y)$ and so $[\langle \overline{x}, x\rangle]I_\alpha = [\langle \overline{x}, y\rangle]$, i.e., $\varphi(x)I_\alpha = \varphi(y)$. Hence, there is an arc of H from $\varphi(x)$ to $\varphi(y)$ coloured by I_α.

Since a transition-preserving bijection between the states of complete automata is necessarily an isomorphism, and the disjoint union of isomorphic Cayley graphs is still a Cayley graph, this completes the proof of Theorem 2. ▮

4 Recognizing Cayley colour graphs

Consider the following problems:

Problem 1 CAYLEY COLOUR GRAPH
Instance: A graph G with colouring λ.
Question: Is G a Cayley colour graph?

Theorem 3 CAYLEY COLOUR GRAPH is in AC^1.

Proof. In [BV96] it has been shown that the analogous problem which asks for G having a weak sense of direction is in AC^1. Theorem 2 then immediately yields the desired result (one has to discard non-outregular graphs and graphs using more colours than the outdegree, but this can be easily done in AC^1). ∎

Problem 2 CAYLEY GRAPH
Instance: A graph G.
Question: Is G a Cayley graph?

A hardness result for CAYLEY GRAPH would have major consequences for most of the problems related to sense of direction. In particular, the problems of establishing whether a graph can be given a (weak,transitive) sense of direction with a given number of colours, or in a minimal way, are at least as hard as CAYLEY GRAPH, even in the case the instances are restricted to outregular graphs. We conjecture that CAYLEY GRAPH is at least as hard as graph isomorphism.

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