On Some Constructions Which Preserve Sense of Direction

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Abstract

Sense of direction is a property of (anonymous) networks which allows to assign coherently local identifiers to other processors on the basis of the route followed by incoming messages. While this property has been in use for quite a long time, it was first defined formally in [FMS94]. In [BV96b] the authors have given a purely combinatorial, graph-theoretical characterization of sense of direction together with efficient algorithms which allow to check if an existing colouring is a sense of direction. However, the problem of establishing a sense of direction using a minimum number of colours has not yet been solved. In this paper we study some constructions which allow to derive large graphs with sense of direction using a small number of colours by exploiting compositional, algebraic and geometric techniques.

1 Introduction

The topological structure of distributed systems can be described by a graph, with nodes representing agents and arcs representing links. Each node has a local (partial) view of the system, and it associates a different label (colour) to each of its incident links.

The solution to many problems in a distributed system can be greatly simplified by using colourings with special properties. In particular, in this paper we study *sense of direction*, a property of global consistency which has been known for a long time (see, for instance, [LMW86, AvLSZ89, San84]), but was not formalized until [FMS94]. In a previous paper [BV96b] the authors characterized weak sense of direction in a combinatorial manner, and showed that it can be decided efficiently in parallel; moreover, they analogously showed that sense of direction is decidable in sequential polynomial time.

A challenging problem is that of finding a colouring which gives (weak) sense of direction to a given graph using a small number of colours. This problem, introduced firstly in [FS96], has not yet been studied in a systematic manner, and we conjecture that it is NP-complete to decide whether a certain number of colours is sufficient to give sense of direction to a given graph.

Another possible point of view is that of creating large "regular" graphs which have sense of direction starting from small ones, trying to waste as few additional colours as possible. In this paper we shall present some results along this line. Some of our constructions are mathematically inspired, while others stem from practical intuitions about modular composition of networks.

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2 Sense of Direction

In this paper we follow the notation of [BV96b], with slight extensions in order to handle multigraphs.

A *(directed) (multi)graph* G is defined by a nonempty set V_G of vertices, a set A_G of arcs, and by two functions $s_G, t_G : A_G \to V_G$ specifying the source and the target of each arc (the subscript G will be omitted whenever no confusion is possible). We write $P[x, y] \subseteq A^*$ for the set of paths from the vertex x to the vertex y. Note that, as done in [BV96b], we use an arc from x to y to represent a link through which y can send messages to x; while this choice could seem counterintuitive, it highly simplifies our proofs.

A *colouring* of a graph G is a function $\lambda : A \to \mathscr{L}$, where $\mathscr{L}$ is a finite set of colours. We say that λ is *deterministic* iff for any pair of arcs a, b such that $s(a) = s(b)$ we have

$$\lambda(a) = \lambda(b) \implies t(a) = t(b),$$

i.e., if the automaton described by the transition graph G with colouring λ is deterministic. We shall frequently shift between graph-theoretic and automata-theoretic concepts when talking about a coloured graph.

Given a graph G deterministically coloured by λ (we shall omit in the future to mention that λ is deterministic; all colourings in this paper are such), let

$$L(x, y) = \{\lambda^*(\pi) \mid \pi \in P[x, y]\},$$

where $\lambda^* : A^* \to \mathscr{L}^*$ is the pointwise lifting of λ. In other words, $L(x, y)$ is the *language* recognized by G when x is the initial state and y is the final state. For all $I \subseteq V^2$ let

$$L_I = \bigcup_{\langle x,y\rangle \in I} L(x, y)$$

(of course, $L(x, y) = L_{\{\langle x,y\rangle\}}$). Notice that $\varepsilon \in L(x, x) \neq \varnothing$.

A *local naming* for G is a family of injective functions $\beta = \{\beta_x : V \to \mathscr{N}\}_{x \in V}$, with $\mathscr{N}$ a finite set, called the *name space*. Intuitively, each node x of G gives to each other node y a name $\beta_x(y)$ taken from the name space. Since we require injectivity, we have that necessarily $|\mathscr{N}| \geq |V|$. We shall also write $\beta : V \times V \to \mathscr{N}$ for the "unindexed" local naming.

Given a coloured graph endowed with a local naming, a function $f : L_{V^2} \to \mathscr{N}$ is a *coding function* iff

$$\forall x, y \in V \quad \forall \pi \in P[x, y] \qquad f(\lambda^*(\pi)) = \beta_x(y).$$

A coding function translates the colouring of the path along which two nodes x, y are connected into the name which x gives to y. Note that while the resulting name is *local* (i.e., x and z might choose different element of the name space for the same node y), the coding function is *global* (i.e., it is the same for all nodes).

A colouring λ is a *weak sense of direction* for a graph G iff for some local naming there is a coding function. We shall also say that a coloured graph *has* weak sense of direction, or that λ *gives* weak sense of direction to G.

Finally, a *decoding function* is a map $h : \mathscr{L} \times \mathscr{N} \to \mathscr{N}$ which satisfies

$$\forall a \in A \quad \forall z \in V \quad \forall \pi \in P[t(a), z] \qquad h(\lambda(a), \beta_{t(a)}(z)) = \beta_{s(a)}(z).$$

A decoding function translates the name given by $t(a)$ to z into the name given by $s(a)$ to z knowing only the colour of the arc a.

A colouring λ is a *sense of direction* for a graph G iff for some local naming there is a coding function and a decoding function.

3 Some basic considerations

In this section we gather some straightforward definitions and results which we shall use implicitly in the rest of the paper. The constructions described here do preserve sense of direction, but in a way which is not advantageous with respect to the number of colours used. We start by introducing a special kind of sense of direction which we shall need in Section 5.

Definition 1 A weak sense of direction is *transitive* if its name space is a multiplicative group, and $\beta_x(y) \cdot \beta_y(z) = \beta_x(z)$, for all vertices x, y and z.

A trivial example of transitive sense of direction is given by Cayley graphs [FS96], and by the coordinate, polar and chordal senses of directions [FMS94]. Note that a much weaker condition must, in fact, be checked:

Proposition 1 Let λ be a weak sense of direction. If β is surjective and the name space $\mathscr{N}$ has a semigroup structure satisfying $\beta_x(y) \cdot \beta_y(z) = \beta_x(z)$, then λ is transitive.

Proof. The unit of $\mathscr{N}$ is given by $\beta_x(x)$ (which is the same for all x). Indeed, $\beta_x(x) \cdot \beta_x(y) = \beta_x(y)$ for all x and y. For the same reason, each $\beta_x(y)$ has inverse $\beta_y(x)$. ▮

We prove immediately that

Theorem 1 A transitive weak sense of direction is a sense of direction.

Proof. Just define a decoding function h by

$$h(\sigma, \beta_y(z)) = f(\sigma) \cdot \beta_y(z) = \beta_x(z),$$

where σ colours an arc from x to y. ▮

Theorem 1 cannot be inverted. In fact, the following counterexample shows a graph which possesses sense of direction, as the reader can easily check using the techniques described in [BV96b]:

$$\begin{tikzcd}
 & 2 & & 5 & 6 \arrow[d, bend right, "b"'] \arrow[d, bend left, "a"] & 8 \arrow[r, "e"] \arrow[d, "d"'] & 10 \\
0 \arrow[r, "a"'] \arrow[ur, "d"] & 1 \arrow[u, "c"'] & 3 \arrow[r, "b"'] \arrow[ur, "e"] & 4 \arrow[u, "c"'] & 7 & 9 &
\end{tikzcd}$$

However, this graph does not possess a *transitive* sense of direction. In order to prove this fact, it is sufficient to look at the following chain of equalities, which contradicts the injectivity of any transitive local naming β, with coding function f, for the previous graph:

$$\beta_8(9) = f(d) = \beta_0(2) = \beta_0(1) \cdot \beta_1(2) = f(a) \cdot f(c) = \beta_6(7) \cdot \beta_4(5) = \\ f(b) \cdot \beta_4(5) = \beta_3(4) \cdot \beta_4(5) = \beta_3(5) = f(e) = \beta_8(10).$$

Proposition 2 If G has (weak,transitive) sense of direction then any (partial) subgraph of G has (weak,transitive) sense of direction.

Proposition 3 If G has (weak) sense of direction then the graph obtained by adding to G an arc with a brand new colour has (weak) sense of direction.

An immediate corollary is that one can change the colour of some arcs of a graph to brand new colours without losing (weak) sense of direction.

As a last point, we show how the product of two graphs having sense of direction can be trivially endowed with a sense of direction using the cartesian product of the colour and name spaces of the two original graphs. The *product* $G \times H$ of two graphs G, H has vertex set $V_G \times V_H$ and arc set $A_G \times A_H$, with $s(\langle a, b\rangle) = \langle s(a), s(b)\rangle$ and $t(\langle a, b\rangle) = \langle t(a), t(b)\rangle$ for all $a \in A_G$ and $b \in A_H$, i.e., there is an arc between $\langle x, y\rangle$ and $\langle x', y'\rangle$ whenever there is an arc between x and x' in G, and an arc between y and y' in H.

Theorem 2 Let λ, μ be (weak,transitive) senses of direction for G and H, respectively. Then $\lambda \times \mu$ is a (weak,transitive) sense of direction for $G \times H$.

Proof. Let f and g ($\mathscr{N}$ and $\mathscr{M}$, β and γ) be the coding functions (name spaces, local namings) of G and H, respectively. Clearly,

$$c(\langle\sigma_1, \tau_1\rangle\langle\sigma_2, \tau_2\rangle \cdots \langle\sigma_n, \tau_n\rangle) = \langle f(\sigma_1\sigma_2 \cdots \sigma_n), g(\tau_1\tau_2 \cdots \tau_n)\rangle$$

defines a coding function for $G \times H$ with local naming $\delta_{\langle x,y\rangle}(\langle x', y'\rangle) = \langle\beta_x(x'), \gamma_y(y')\rangle$. Indeed, for all $\pi \in P[\langle x, y\rangle, \langle x', y'\rangle]$ we have

$$c((\lambda \times \mu)^*(\pi)) = c((\lambda \times \mu)^*(\langle a_1, b_1\rangle\langle a_2, b_2\rangle \cdots \langle a_n, b_n\rangle)) = \\ c(\langle\lambda(a_1), \mu(b_1)\rangle \cdots \langle\lambda(a_n), \mu(b_n)\rangle) = \langle f(\lambda^*(a_1a_2 \cdots a_n)), g(\mu^*(b_1b_2 \cdots b_n))\rangle = \\ \langle\beta_x(x'), \gamma_y(y')\rangle = \delta_{\langle x,y\rangle}(\langle x', y'\rangle).$$

Analogously, it is easy to check that

$$\ell(\langle\lambda(a), \mu(b)\rangle, \delta_{\langle t(a),t(b)\rangle}(\langle z, z'\rangle)) = \langle h(\lambda(a), \beta_{t(a)}(z)), k(\mu(b), \gamma_{t(b)}(z'))\rangle$$

is a decoding function for $G \times H$ whenever h and k are decoding functions for G and H, respectively. Finally, since δ is defined componentwise, the transitive case follows trivially. ∎

4 Regular Coverings

In this section we study how *regular graph coverings* [GT87], i.e., graph coverings induced by a suitable action of a group, have sense of direction exactly when the graph they cover has. This fact formalizes the intuition that a large graph which is "locally like" a small graph with sense of direction and has a high degree of internal symmetry has in fact sense of direction. As a very particular case we obtain the fact that Cayley graphs possess transitive sense of direction.

Coverings of undirected graphs are well known in topology and graph theory (see, for instance, [Big74]). We generalize this definition to directed graphs, using the notion of (op)fibration of graphs introduced in [BV96a], and used in [BCG$^+$96] in order to characterize anonymous networks in which election is possible. We briefly illustrate the geometric meaning of the notion of fibration, in order to make the intuition behind the next definition clear. Recall that a *graph morphism* $G \to H$ is given by a pair of functions (ambiguously denoted with the same name) $\varphi : V_G \to V_H$ and $\varphi : A_G \to A_H$ which commute with the source and target functions, i.e., $s \circ \varphi = \varphi \circ s$ and $t \circ \varphi = \varphi \circ t$. In other words, a morphism maps nodes to nodes and arcs to arcs in such a way to preserve the incidence relation. In the coloured case, we also require the map on the arcs to preserve colours. Then we have the following

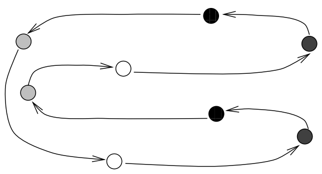
Definition 2 A *fibration* between graphs G and H is a morphism $\varphi : G \to H$ such that for each arc $a \in A_H$ and for each $x \in V_G$ such that $\varphi(x) = t(a)$ there is a unique $\widetilde{a}^x \in A_G$ such that $\varphi(\widetilde{a}^x) = a$ and $t(\widetilde{a}^x) = x$.

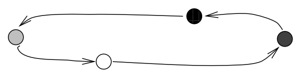
There is a very intuitive characterization of fibrations based on the concept of local isomorphism. A fibration $\varphi : G \to H$ induces an equivalence relation between the vertices of G, whose classes are precisely the fibres of φ, i.e., the sets $\varphi^{-1}(x)$. When two vertices x and y are equivalent (i.e., they are in the same fibre), there is a one-to-one correspondence between arcs ending in x and arcs ending in y which preserves colours, and such that the sources of any two related arcs are equivalent.

A covering projection is a special kind of fibration, where each arc can also be lifted uniquely from its tail; more formally:

Definition 3 A fibration $\varphi : G \to H$ is a *covering projection* if, for every arc $a \in A_H$ and every vertex $x \in V_G$ such that $\varphi(x) = s(a)$, there is a unique ${}^x\widetilde{a} \in A_G$ such that $\varphi({}^x\widetilde{a}) = a$ and $s({}^x\widetilde{a}) = x$. In this case, G is said to be a *covering* of H.

In the case of coverings, the local isomorphism property gives a bijective correspondence between the whole (disjoint) neighbourhoods of two nodes in the same fibre. Moreover, the cardinality of all fibres is the same. In the following picture, we show graphically how an eightcycle covers a four-cycle:





However, the general notion of covering of directed (coloured) graphs is too general, and does not preserve sense of direction. In fact, a general covering is essentially built by taking k copies of G and by permuting the targets of the k copies of each arc. We need some structure in order to make this permutation "uniform" with respect to the given colouring, and this intuitive idea is formalized by using the notion of *regular covering* [GT87]. A regular covering is a covering whose fibres are the orbits of a subgroup of $\mathbf{Aut}(G)$ which acts freely[1] on G.

Our interest is in obtaining certain regular coverings starting from a given base.

Definition 4 Let G be a graph with a colouring $\lambda : A \to \mathscr{L}$, $\mathfrak{G}$ a group and $\xi : \mathscr{L} \to \mathfrak{G}$. The *uniform covering* of G with respect to ξ is the graph G^ξ having set of vertices $V \times \mathfrak{G}$ and set of arcs $A \times \mathfrak{G}$, with $s(\langle a, \mathfrak{g}\rangle) = \langle s(a), \mathfrak{g}\rangle$ and $t(\langle a, \mathfrak{g}\rangle) = \langle t(a), \mathfrak{g} \cdot \xi(\lambda(a))\rangle$.

It is easy to see that the projection $V \times \mathfrak{G} \to V$ is a regular covering projection (induced by the automorphisms $\sigma_\mathfrak{g}$ associated to the elements of $\mathfrak{G}$: $\sigma_\mathfrak{g}(\langle s(a), \mathfrak{h}\rangle) = \langle s(a), \mathfrak{g} \cdot \mathfrak{h}\rangle$). The construction we have just given is essentially that of a *voltage graph* [GT87], but the "voltage assignment map", which is defined on the arcs of G, must in our case depend on the arc colour *only*.

Theorem 3 Let λ be a (weak,transitive) sense of direction for G. Then $\mu(\langle a, \mathfrak{g}\rangle) = \lambda(a)$ is a (weak,transitive) sense of direction for any uniform covering of G.

Proof. Let f, $\mathscr{N}$ and β be the coding functions, names space and local naming of G. Let G^ξ be a uniform covering of G with generating group $\mathfrak{G}$; we define its name space as $\mathscr{N} \times \mathfrak{G}$, and its local naming γ as

$$\gamma_{\langle x, \mathfrak{g}\rangle}(\langle y, \mathfrak{h}\rangle) = \langle \beta_x(y), \mathfrak{g}^{-1} \cdot \mathfrak{h}\rangle;$$

then, the following function

$$c(\sigma_1\sigma_2 \cdots \sigma_n) = \langle f(\sigma_1\sigma_2 \cdots \sigma_n), \xi(\sigma_1) \cdot \xi(\sigma_2) \cdot \cdots \cdot \xi(\sigma_n)\rangle$$

is a coding function for G^ξ. Indeed, suppose $\langle a_1, \mathfrak{g}_1\rangle\langle a_2, \mathfrak{g}_2\rangle \cdots \langle a_n, \mathfrak{g}_n\rangle$ is a path from $\langle x, \mathfrak{g}_1\rangle$ to $\langle y, \mathfrak{g}_{n+1}\rangle$. Then, necessarily $\mathfrak{g}_{i+1} = \mathfrak{g}_i \cdot \xi(\lambda(a_i))$ ($i = 1, 2, \ldots, n$), so

$$c(\lambda^*(a_1a_2 \cdots a_n)) = \langle f(\lambda^*(a_1a_2 \cdots a_n)), \xi(\lambda(a_1))\xi(\lambda(a_2)) \cdots \xi(\lambda(a_n))\rangle =$$
$$\langle \beta_x(y), \mathfrak{g}_1^{-1} \cdot \mathfrak{g}_2 \cdot \mathfrak{g}_2^{-1} \cdot \mathfrak{g}_3 \cdot \cdots \cdot \mathfrak{g}_n^{-1} \cdot \mathfrak{g}_{n+1}\rangle = \langle \beta_x(y), \mathfrak{g}_1^{-1} \cdot \mathfrak{g}_{n+1}\rangle = \gamma_{\langle x, \mathfrak{g}_1\rangle}(\langle y, \mathfrak{g}_{n+1}\rangle).$$

[1]A subgroup $\Sigma \subseteq \mathbf{Aut}(G)$ acts *freely* if every element of Σ (except the identity) has no fixpoints.

If we also have a decoding function h for G, then a decoding function k for G^{ξ} can be defined as follows:

$$k(\sigma, \langle \nu, \mathfrak{g} \rangle) = \langle h(\sigma, \nu), \xi(\sigma) \cdot \mathfrak{g} \rangle,$$

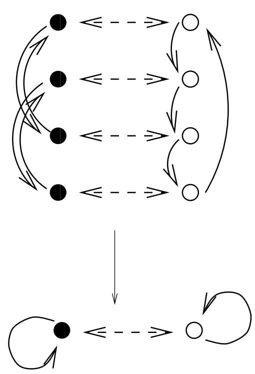
where $\nu \in \mathscr{N}$. In order to complete the argument, we show that k is indeed a decoding function; suppose $\langle a, \mathfrak{g} \rangle$ is an arc from $\langle x, \mathfrak{g} \rangle$ to $\langle y, \mathfrak{h} \rangle$ (where $\mathfrak{h} = \mathfrak{g} \cdot \xi(\lambda(a))$)

$$k(\mu(\langle a, \mathfrak{g} \rangle), \gamma_{\langle y, \mathfrak{h} \rangle}(\langle z, \mathfrak{k} \rangle)) = k(\lambda(a), \langle \beta_y(z), \mathfrak{h}^{-1} \cdot \mathfrak{k} \rangle) =$$
$$\langle h(\lambda(a), \beta_y(z)), \xi(\lambda(a)) \cdot \mathfrak{h}^{-1} \cdot \mathfrak{k} \rangle = \langle \beta_x(z), \xi(\lambda(a)) \cdot (\xi(\lambda(a)))^{-1} \cdot \mathfrak{g}^{-1} \cdot \mathfrak{k} \rangle =$$
$$\langle \beta_x(z), \mathfrak{g}^{-1} \cdot \mathfrak{k} \rangle = \gamma_{\langle x, \mathfrak{g} \rangle}(\langle z, \mathfrak{k} \rangle).$$

Finally, if the given sense of direction is transitive we have that $\mathscr{N} \times \mathfrak{G}$ is a monoid with componentwise multiplication, and

$$\gamma_{\langle x, \mathfrak{g} \rangle}(\langle y, \mathfrak{h} \rangle) \cdot \gamma_{\langle y, \mathfrak{h} \rangle}(\langle z, \mathfrak{k} \rangle) = \langle \beta_x(y), \mathfrak{g}^{-1} \cdot \mathfrak{h} \rangle \cdot \langle \beta_y(z), \mathfrak{h}^{-1} \cdot \mathfrak{k} \rangle =$$
$$\langle \beta_x(y) \cdot \beta_y(z), \mathfrak{g}^{-1} \cdot \mathfrak{h} \cdot \mathfrak{h}^{-1} \cdot \mathfrak{k} \rangle = \langle \beta_x(z), \mathfrak{g}^{-1} \cdot \mathfrak{k} \rangle = \gamma_{\langle x, \mathfrak{g} \rangle}(\langle z, \mathfrak{k} \rangle). \blacksquare$$

In order to see that a regular projection which is *not* uniform does not preserve sense of direction, consider the regular projection shown in the following picture:



The upper graph clearly does not have weak sense of direction (just check the different lengths of the cycles on the right and on the left). Yet, it is obtained from the lower graph (which has transitive sense of direction) by assigning an element of $\mathbf{Z}_4$ to each arc. This produces a regular covering (in fact, we used a voltage assignment in the sense of [GT87]), but not a uniform covering, because both self-loops have the same colour, but they have been assigned different elements of $\mathbf{Z}_4$.

We now derive some interesting consequences of the previous theorem. Recall that the *Cayley graph* of a group $\mathfrak{G}$ with respect to a set of generators $S = \{\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n\}$ has vertex set $\mathfrak{G}$, arc set $\mathfrak{G} \times S$ and $s(\langle \mathfrak{g}, \mathfrak{g}_k \rangle) = \mathfrak{g}$, $t(\langle \mathfrak{g}, \mathfrak{g}_k \rangle) = \mathfrak{g} \cdot \mathfrak{g}_k$.

Corollary 1 The colouring $\lambda(\langle \mathfrak{g}, \mathfrak{g}_k \rangle) = \mathfrak{g}_k$ gives transitive sense of direction to any Cayley graph.

Proof. Just choose G to be the one-point graph endowed with $|S|$ many self-loops, $\mathfrak{G}$ the generating group, $\xi : S \to \mathfrak{G}$ the injection and apply Theorem 3. ▮

This result had been previously proved in [FS96]. Theorem 3 exhibits uniform coverings as a wide generalization of Cayley graphs, in which sense of direction is guaranteed *locally* by the structure of G, and *globally* by the structure of the generating group.

Next, we consider the problem of *minimality*. A graph has *minimal (weak, transitive) sense of direction* [FS96] iff it can be given (weak, transitive) sense of direction using as many colours as its maximum outdegree.

Corollary 2 G has minimal (weak, transitive) sense of direction iff $G \cong H^\xi$ for some coloured graph H with minimal (weak, transitive) sense of direction.

Proof. The "only if" part is trivial (take $G = H$). In view of Theorem 3, the other direction requires just showing that the outdegree of H^ξ is the same of H, which is true by the local isomorphism property of covering projections. ▮

One is usually interested in graphs which are strongly connected. The following theorem characterizes the maps ξ which produce strongly connected uniform coverings.

Theorem 4 Let G be strongly connected. Then G^ξ is strongly connected iff $\hat{\xi}(L(x, y)) = \mathfrak{G}$ for[2] all pairs $x, y \in V_G$.

Proof. Observe that

$$L(\langle x, \mathfrak{g}\rangle, \langle y, \mathfrak{h}\rangle) = L(x, y) \cap \{w \in \mathscr{L}^* \mid \mathfrak{h} = \mathfrak{g} \cdot \hat{\xi}(w)\}.$$

Thus, if we fix x and y and let $\mathfrak{g} = 1$, $L(\langle x, 1\rangle, \langle y, \mathfrak{h}\rangle) \neq \varnothing$ for all $\mathfrak{h}$ implies that for each $\mathfrak{h}$ there is a path π from x to y in G such that $\hat{\xi}(\pi) = \mathfrak{h}$, so that $\hat{\xi}(L(x, y)) = \mathfrak{G}$. On the other hand, given $\langle x, \mathfrak{g}\rangle$ and $\langle y, \mathfrak{h}\rangle$, $L(x, y)$ contains a string w such that $\hat{\xi}(w) = \mathfrak{g}^{-1} \cdot \mathfrak{h}$. Let $\pi = a_1 a_2 \cdots a_n$ be the path whose colour string is w. Then the path

$$\langle a_1, \mathfrak{g}\rangle\langle a_2, \mathfrak{g} \cdot \xi(\lambda(a_1))\rangle \cdots \langle a_n, \mathfrak{g} \cdot \xi(\lambda(a_1)) \cdot \xi(\lambda(a_2)) \cdot \cdots \cdot \xi(\lambda(a_{n-1}))\rangle$$

starts from $\langle x, \mathfrak{g}\rangle$ and ends on

$$\langle y, \mathfrak{g} \cdot \xi(\lambda(a_1)) \cdot \xi(\lambda(a_2)) \cdot \cdots \cdot \xi(\lambda(a_n))\rangle = \langle y, \mathfrak{g} \cdot \hat{\xi}(w))\rangle = \langle y, \mathfrak{g} \cdot \mathfrak{g}^{-1} \cdot \mathfrak{h}\rangle = \langle y, \mathfrak{h}\rangle. ▮$$

The previous theorem requires checking n^2 equivalences, where n is the number of nodes of G. We now show how to reduce this requirement.

Corollary 3 Let G be strongly connected. Then G^ξ is strongly connected iff for all $x \in V_G$ we have $\hat{\xi}(L(x, x)) = \mathfrak{G}$.

[2] We denote with $\hat{\xi}$ the map $\mathscr{L}^* \to \mathfrak{G}$ induced by the free monoid property, i.e., $\hat{\xi}(\sigma_1\sigma_2 \cdots \sigma_n) = \xi(\sigma_1) \cdot \xi(\sigma_2) \cdot \cdots \cdot \xi(\sigma_n)$.

Proof. Let $x, y \in V$ and π a path from x to y. Then, by hypothesis, when ρ ranges over $P[y, y]$ (i.e., ρ is an oriented cycle passing through y) $\hat{\xi}(\lambda^*(\rho))$ covers all of $\mathfrak{G}$, and consequently the same happens for $\hat{\xi}(\lambda^*(\rho\pi))$. This means that $\hat{\xi}(L(x, y)) \supseteq \hat{\xi}(\lambda^*(\pi)) \cdot \hat{\xi}(L(y, y)) = \mathfrak{G}$. ▮

It is interesting to note that, in fact, one needs just to check that the conditions of the previous theorem are true along the arcs of an arbitrary spanning subgraph[3] of G:

Corollary 4 Let G be strongly connected, and H a spanning subgraph of G. Then G^{ξ} is strongly connected iff for all arcs a of H we have $\hat{\xi}(L(s(a), t(a))) = \mathfrak{G}$.

Proof. We just need to show that for all $x \in V_G$ we have $\hat{\xi}(L(x, x)) = \mathfrak{G}$. As in the proof of the previous corollary, assuming without loss of generality that $x = s(a)$ for an arc a of H, there is certainly a path π going from $y = t(a)$ to x, which can be prefixed by path ρ going from x to y. When ρ varies through $P[x, y]$, $\hat{\xi}(\lambda^*(\rho))$ varies through all of $\mathfrak{G}$, and consequently the same happens for $\hat{\xi}(\lambda^*(\rho\pi))$. ▮

5 Graph Decompositions

In the current literature, there are many known methods for colouring a given graph in such a way that it has sense of direction (see, e.g., [FMS94]), but all of them are essentially "global" in their nature, in the sense that they build the colouring from scratch; moreover, the number of colours used in those constructions is in general not minimal.

An interesting question is whether one can build large networks with sense of direction by connecting smaller ones, using a limited number of additional colours.

In this section we propose a notion of graph decomposition which allows one to build complex graphs starting from simple components, in much the same way one is used to build large networks starting from smaller subnets, and using a certain higher-level topology in order to connect them.

The basic idea behind our construction is that it should be possible to break the route followed by a message into *local* and *global* displacements, which are distinguished by a bipartition on the set of colours. Each subnet, as well as the "global connection" network (once the single subnets are collapsed to points), should be endowed with a sense of direction. This leads us to the following

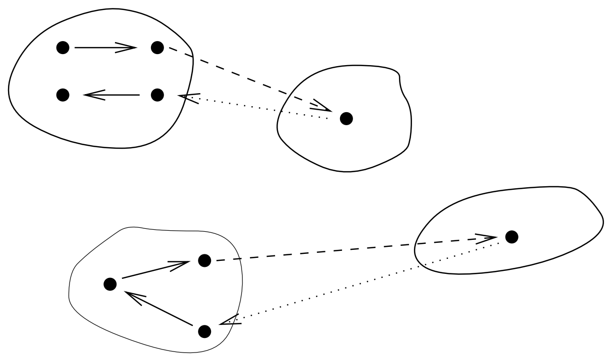
Definition 5 Let G be a graph coloured on the set[4] $\mathscr{L} + \mathscr{M}$, N a graph coloured on $\mathscr{L}$ (called the *local topology*) and S a graph coloured on $\mathscr{M}$ (called the *skeleton*). An $\langle S, N\rangle$-*decomposition* of G is given by a partition of the vertex set V_G into sets $V_1, V_2, \ldots\ V_n$ such that the subgraphs induced by the V_i's (called *subnets*) are each isomorphic to some subgraph of N, and the graph obtained by substituting in G each subnet with a single vertex is isomorphic to S.

[3] We recall that a *spanning subgraph* of a graph G is given by a subset of the arcs of G which covers all of its nodes, i.e., for all $x \in V$ either $x = s(a)$ or $x = t(a)$ for some arc a of H.

[4] With the plus sign we denote disjoint union.

Note that the conditions of the previous definition imply that there is no $\mathscr{L}$-coloured arc between nodes which live in different subnets. In what follows we shall denote the isomorphism between subnets and subgraphs of N with an overline; thus, the name of a vertex $x \in V_i$ as a node of N will be $\overline{x}$.

The basic problem we have to deal with in order to endow G with a sense of direction, however, is given by the interaction between local naming and movement along global connections. In fact, an unconstrained global movement between nodes which have a different local view could "scramble" the information carried by the respective local namings in such a way to make a translation between the local names impossible. The following example shows how this could happen: the global connections create two paths with the same labels, but only one of them is a loop (which contradicts the existence of weak sense of direction).



These considerations lead us to the following

Theorem 5 Consider an $\langle S, N\rangle$-decomposition of G with colouring $\lambda : A \to \mathscr{L}+\mathscr{M}$. If N has transitive sense of direction (with name space $\mathscr{N}$ and local naming β), S has (weak,transitive) sense of direction and there is a function $d : \mathscr{M} \to \mathscr{N}$ such that $\beta_{\overline{s(a)}}(\overline{t(a)}) = d(\lambda(a))$ for all $\mathscr{M}$-coloured arcs a, then G has (weak,transitive) sense of direction.

Proof. Let f, $\mathscr{N}$, β and g, $\mathscr{P}$, γ be the coding functions, name spaces and local namings of N and S, respectively. The name space of G will be given by $\mathscr{P} \times \mathscr{N}$, and the local naming δ by

$$\delta_x(y) = \langle \gamma_i(j), \beta_{\overline{x}}(\overline{y})\rangle$$

where $x \in V_i$, $y \in V_j$ (a straightforward check shows that δ_x is injective). The coding function is then given by

$$c(v_1 w_1 v_2 w_2 \cdots v_n w_n) = \langle g(v_1 v_2 \cdots v_n), \hat{d}(v_1) \cdot f(w_1) \cdot \hat{d}(v_2) \cdot f(w_2) \cdot \cdots \cdot \hat{d}(v_n) \cdot f(w_n)\rangle,$$

where $v_i \in \mathscr{M}^*$ while $w_i \in \mathscr{L}^*$. This is a coding function; indeed, let $\pi_1\rho_1\pi_2\rho_2 \cdots \pi_n\rho_n$ be a path from x to y such that the π_i's are $\mathscr{M}$-coloured and the ρ_i's are $\mathscr{L}$-coloured, all of them being non-empty (the other cases can be proved analogously), and let π_i go from x_i to y_i (so $x_1 = x$). Then

$$\begin{aligned} c(\lambda^*(\pi_1\rho_1\pi_2\rho_2 \cdots \pi_n\rho_n)) &= c(\lambda^*(\pi_1)\lambda^*(\rho_1) \cdots \lambda^*(\pi_n)\lambda^*(\rho_n)) = \\ &\langle g(\lambda^*(\pi_1 \cdots \pi_n)), \hat{d}(\lambda^*(\pi_1)) \cdot f(\lambda^*(\rho_1)) \cdot \cdots \cdot \hat{d}(\lambda^*(\pi_n)) \cdot f(\lambda^*(\rho_n))\rangle = \\ &\quad \langle \gamma_i(j), \beta_{\overline{x_1}}(\overline{y_1}) \cdot \beta_{\overline{y_1}}(\overline{x_2}) \cdot \cdots \cdot \beta_{\overline{x_n}}(\overline{y_n}) \cdot \beta_{\overline{y_n}}(\overline{y})\rangle = \langle \gamma_i(j), \beta_{\overline{x}}(\overline{y})\rangle = \delta_x(y). \end{aligned}$$

We used the fact that for each π_i we have $\hat{d}(\lambda^*(\pi_i)) = \beta_{\overline{x_i}}(\overline{y_i})$ by the transitive property of β.

In case a decoding function h exists for S, we may define a decoding function for G by

$$\ell(\sigma, \langle \eta, \nu \rangle) = \begin{cases} \langle h(\sigma, \eta), d(\sigma) \cdot \nu \rangle & \text{if } \sigma \in \mathscr{M} \\ \langle \eta, f(\sigma) \cdot \nu \rangle & \text{if } \sigma \in \mathscr{L}. \end{cases}$$

We examine the two possible cases in order to prove that ℓ is indeed a decoding function. Let a be an arc from $x \in V_i$ to $y \in V_j$, $\lambda(a) = \sigma \in \mathscr{M}$ and $z \in V_k$. Then

$$\ell(\sigma, \delta_y(z)) = \ell(\sigma, \langle \gamma_j(k), \beta_{\overline{y}}(\overline{z}) \rangle) = \langle h(\sigma, \gamma_j(k)), d(\sigma) \cdot \beta_{\overline{y}}(\overline{z}) \rangle = \\ \langle h(\sigma, \gamma_j(k)), \beta_{\overline{x}}(\overline{y}) \cdot \beta_{\overline{y}}(\overline{z}) \rangle = \langle \gamma_i(k), \beta_{\overline{x}}(\overline{z}) \rangle = \delta_x(z).$$

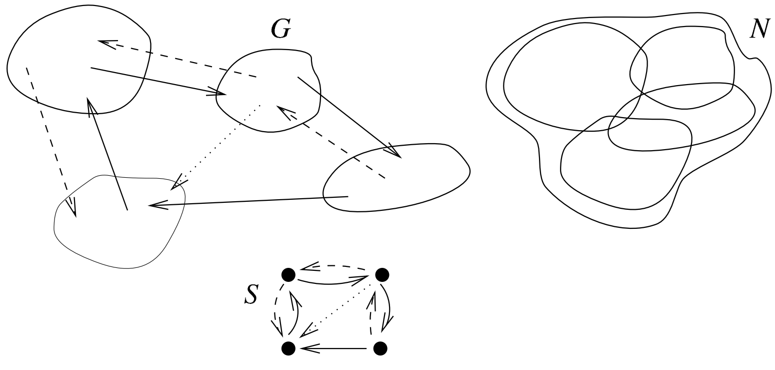
On the other hand, if $\sigma \notin \mathscr{M}$

$$\ell(\sigma, \delta_y(z)) = \ell(\sigma, \langle \gamma_j(k), \beta_{\overline{y}}(\overline{z}) \rangle) = \langle \gamma_j(k), f(\sigma) \cdot \beta_{\overline{y}}(\overline{z}) \rangle = \\ \langle \gamma_j(k), \beta_{\overline{x}}(\overline{y}) \cdot \beta_{\overline{y}}(\overline{z}) \rangle = \langle \gamma_j(k), \beta_{\overline{x}}(\overline{z}) \rangle = \delta_x(z).$$

Finally, if the sense of direction of S is transitive we can define a monoid operation on $\mathscr{P} \times \mathscr{N}$ componentwise, and then

$$\delta_x(y) \cdot \delta_y(z) = \langle \gamma_i(j), \beta_{\overline{x}}(\overline{y}) \rangle \cdot \langle \gamma_j(k), \beta_{\overline{y}}(\overline{z}) \rangle = \\ \langle \gamma_i(j) \cdot \gamma_j(k), \beta_{\overline{x}}(\overline{y}) \cdot \beta_{\overline{y}}(\overline{z}) \rangle = \langle \gamma_i(k), \beta_{\overline{x}}(\overline{z}) \rangle = \delta_x(z). \blacksquare$$

The situation can be sketched as in the following picture, where the small blobs represent subnets:



A very simple example is given by a set of hypercubes (not necessarily of constant dimension) connected in a ring topology. In this case the number of colours used by the decomposition theorem is the maximum dimension of a hypercube plus two (in fact, this number is trivially minimal).

6 Conclusions and Open Problems

In this paper we have described some techniques which allow one to build large graphs with an "inexpensive" sense of direction, i.e., using a small number of colours. Clearly, this number is always bounded from below by the maximum degree of the graph (counting each neighbour just once), and from above by the number of vertices (order arbitrarily the vertices and then assign to each arc from the i-th to the j-th vertex the colour $(i - j) \bmod n$, n being the number of vertices). In the case of Cayley graphs, the lower bound is achieved; in the case of a complete graph (with self-loops) the two bounds coincide.

The general problem (given a graph G and an integer k, can G be given a sense of direction using at most k colours?) remains open, and challenging. Since this problem is likely to be difficult, and possibly NP-complete, we think it is interesting to develop a set of techniques that allow to build (or to prove the existence of) a sense of direction using a number of primitive constructions. The results of this paper should be seen as a first step in this direction.

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