Dynamic Monopolies in Tori *

Paola Flocchini † Elena Lodi ‡ Fabrizio Luccio §
Linda Pagli‡ Nicola Santoro¶

Abstract

Let G be a simple connected graph where every node is colored either *black* or *white*. Consider now the following repetitive process on G: each node recolors itself, at each local time step, with the color held by the majority of its neighbors. Depending on the initial assignment of colors to the nodes and on the definition of majority, different *dynamics* can occur. We are interested in *dynamos*; i.e., initial assignments of colours which lead the system to a monocromatic configuration in a finite number of steps. In the context of distributed computing and communication networks, this repetitive process is particularly important in that it describes the impact that a set of initial faults can have in majority-based systems (where black nodes correspond to faulty elements and white to non-faulty ones). In this paper we study two particular forms of dynamos (*irreversible* and *monotone*) in *tori*, focusing on the minimum number of initial black elements needed to reach the fixed point. We derive lower and upper bounds on the size of dynamos for three types of tori, under different assumptions on the majority rule (*simple* and *strong*). These bounds are *tight* within an additive constant. The upper bounds are *constructive*: for each topology and each majority rule, we exhibit a dynamo of the claimed size.

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†School of Information Technology and Engineering, University of Ottawa, Canada. (flocchin@site.uottawa.ca)

‡Dipartimento di Matematica, Università di Siena, Italy. (lodi@di.unipi.it)

§Dipartimento di Informatica, Università di Pisa, Italy. ({luccio, pagli}@di.unipi.it)

¶School of Computer Science, Carleton University, Ottawa, Canada. (santoro@scs.carleton.ca)

1 Introduction

In *majority-based* distributed systems and communication networks, faulty elements can induce a faulty behavior in their neighbors.

This is for example the situation in distributed systems where majority voting among various copies of crucial data are performed between neighbours at each step [18]: if the majority of its neighbors is faulty (i.e., has corrupted data), a non-faulty element will exhibit a faulty behavior (i.e., its data will become corrupted) and will therefore be indistinguishable from a faulty one. For such systems, the dynamics of the fault propagation describes then the behaviour of the systems. Let G be the graph describing the communication topology of the system; then the dynamics of the system can be viewed as a repetitive process on G: initially, each node is colored either *black* (faulty) or *white* (non-faulty); every node recolors itself at each local time step, according to the color of the "majority" of its neighbors. This process is possibly asynchronous, as the local clocks might not be synchronized; in this case the global time unit is the one of the slowest node.

The dynamics of majority-based systems have been extensively studied in the context of cellular automata, where effort has been concentrated on determining the asymptotic behaviors of different majority rules on different graph structures, focusing on the study of configurations leading to *periodic* dynamics on finite graphs [10, 21], on the number of fixed points on finite rings [1, 11] and (finite and infinite) lines [14, 15], and on the behaviors of infinite graphs [16].

Recently, the dynamics of majority-based systems has also been investigated from the distributed computing community. The interest there has been motivated by fault-tolerance and focused almost exclusively on the patterns leading to *monochromatic* fixed points. These patterns, called *dynamos*, correspond to the distributions of initial faults which may lead the entire system to a faulty behaviour, e.g., with every entity holding corrupted data; hence the particular interest.

If the initial faults are permanent, the dynamo is said to be *irreversible*; if the faults can be mended by the majority rule, the dynamo is called *reversible*. A dynamos is *monotonic* if the set of black vertices at any time t is a subset of the one at time $t+1$. Note that irreversible dynamos are always monotone, hence we will use the term "monotone" to indicate "reversible monotone".

Very little is known about dynamos. Their study has first been introduced by Peleg [19]. There have been several studies on *monopolies*, that is monotone dynamos which lead the system to an all black state in a single step [4, 2, 18].

General lower and upper bounds on the size of monotone dynamos have been established in [19], Characterizations of irreversible dynamos of minimal size has been given for chordal rings [6], and butterfly [13]; trade-offs between size and time were recently established [9]. Results on directed graphs were established in the study of *catastrophic fault patterns* for infinite chordal rings (e.g., see [5, 17, 22]) and recently for general digraphs [23].

In this paper we study *irreversible* and *reversible monotone* (or, simply, *monotone*) dynamos in *tori*, focusing on the minimum number of initial black elements needed to reach the fixed point. The torus is one of the simplest and most natural way of connecting processors in a network. We consider three different types of tori: the *toroidal mesh* (the classical architecture used in VLSI), the *torus cordalis* (a.k.a. double-loop interconnection networks), and the *torus serpentinus* (used as the topology of, e.g., ILIAC IV). All these graphs are regular with degree four; hence, two different majority rules are possible to decide whether a node becomes black: *simple* (two neighbours) and *strong* (three neighbours) majority, each leading to different dynamics.[1] These majority systems correspond to bi-dimensional cellular automata with different border rules.

We derive lower and upper bounds on the size of irreversible and monotone dynamos for each of these topologies and for each majority rule. These bounds, summarized in Table 1, are *tight* within an additive constant. The upper bounds are *constructive*. In fact, for each topology and each majority rule, we exhibit a dynamo of the claimed size.

Note that although meshes with toroidal connections avoid the border effects which would occur in meshes, our techniques can be easily adapted to meshes.

2 Basic Definitions

Consider an $m \times n$ mesh. Depending on how the border vertices are linked to other processors we obtain different topologies. The vertices on the last column are usually connected either to the opposite ones on the same rows, thus forming ring connections in each row, or to the opposite ones on the successive rows in a snake-like way. A similar linking strategy can be applied for the last row. In the *toroidal mesh* rings are formed in rows and columns. In the *torus cordalis* there are rings in the columns and snake-like connections in the rows; finally, in the *torus serpentinus* there are snake-like connections in the rows and the columns. Formally, we have:

Definition 1
A toroidal mesh *of $m \times n$ vertices is a mesh where each vertex $\tau_{i,j}$, with $0 \le i \le m-1$, $0 \le j \le n-1$ is connected to the four vertices $\tau_{(i-1) \bmod m, j}$, $\tau_{(i+1) \bmod m, j}$, $\tau_{i,(j-1) \bmod n}$, $\tau_{i,(j+1) \bmod n}$ (mesh connections).*
A torus cordalis *is like a toroidal mesh except that the last vertex $\tau_{i,n-1}$ of each row is connected to the first vertex $\tau_{(i+1) \bmod m, 0}$ of row $i+1$. A* torus serpentinus *is like a toroidal mesh except that the last vertex $\tau_{i,n-1}$ of each row i is connected to the first vertex $\tau_{(i+1) \bmod m, 0}$ of row $i+1$, and the last vertex $\tau_{m-1,j}$ of each column j is connected to the first vertex $\tau_{0,(j-1) \bmod n}$ of column $j-1$.*

Notice that the torus cordalis can be seen as a chordal ring with one chord, and it is known in the literature also as *double loop network* [3].

[1] In the terminology of [19], simple majority and strong majority correspond to the " self-not-included, prefer-black" and "self-not-included, prefer-current", respectively.

Irreversible Dynamos				
	Simple Majority		Strong Majority	
	Lower Bound	*Upper Bound*	*Lower Bound*	*Upper Bound*
Toroidal mesh	$\lceil \frac{m+n}{2} \rceil - 1$	$\lceil \frac{m+n}{2} \rceil - 1$	$\lceil \frac{mn+1}{3} \rceil$	$\lceil \frac{H}{3} \rceil (K+1)$
Torus cordalis	$\lceil \frac{n}{2} \rceil$	$\lfloor \frac{n}{2} \rfloor + 1$	$\lceil \frac{mn+1}{3} \rceil$	$\lceil \frac{m}{3} \rceil (n+1)$
Torus serpentinus	$\lceil \frac{N}{2} \rceil$	$\lfloor \frac{N}{2} \rfloor + 1$	$\lceil \frac{mn+1}{3} \rceil$	$\lceil \frac{H}{3} \rceil (K+1)$

Monotone Dynamos				
	Simple Majority		Strong Majority	
	Lower Bound	*Upper Bound*	*Lower Bound*	*Upper Bound*
Toroidal mesh	$m+n-2$	$m+n-1$	$\lceil \frac{mn+1}{2} \rceil$	$\lceil \frac{mn}{2} + \frac{N}{6} + \frac{2}{3} \rceil^*$
Torus cordalis	$n+1$	$n+1$	$\lceil \frac{mn+1}{2} \rceil$	$\lceil \frac{mn}{2} + \frac{N}{6} + \frac{2}{3} \rceil^*$
Torus serpentinus	$N+1$	$N+1$	$\lceil \frac{mn+1}{2} \rceil$	$\lceil \frac{mn}{2} + \frac{N}{6} + \frac{2}{3} \rceil^*$

Table 1: Bounds on the size of monotone dynamos, for tori of $m \times n$ vertices; $N = min\{m, n\}$, and $H, K = m, n$ or $H, K = n, m$ (choose the alternative that yields stricter bounds). The asterisk denotes a worst case.

Following a standard rule of reversible coloring, each vertex v takes the color of the majority of its neighbours. In case of tie v becomes black (*simple majority*) or mantains its color (*strong majority*). In irreversible coloring the above rule applies only to white vertices, while black vertices remain black. Let $c(v)$ be the color of v, and $N(v)$ be the set of its neighbours. In a torus we have $N(v) = 4$ and the above rules can be formally written as:

- Reversible simple majority

  if $\exists\, x, y \in N(v) : c(x) = c(y) = black$ **then** $c(v) \leftarrow black$ **else** $c(v) \leftarrow white$

- Reversible strong majority

  if $\exists\, x, y, z \in N(v) : c(x) = c(y) = c(z)$ **then** $c(v) \leftarrow c(x)$ **else** $c(v)$ *does not change*

- Irreversible simple majority

  if $\exists\, x, y \in N(v) : c(x) = c(y) = black$ **then** $c(v) \leftarrow black$ **else** $c(v)$ *does not change*

- Irreversible strong majority

  if $\exists\, x, y, z \in N(v) : c(x) = c(y) = c(z) = black$ **then** $c(v) \leftarrow black$ **else** $c(v)$ *does not change*

A *simple* (respectively, *strong*) *white block* W is a subset of T composed of all white vertices, each of which has at least three (repectively two) neighbours in W. For example two adjacent white columns form a simple white block in a toroidal mesh or in a torus cordalis, and a strong white block in a torus serpentinus. White vertices placed on a cycle form a strong white block in any torus. Clearly if a simple (respectively, strong) white block W is present, the subset $T - W$ of the other vertices cannot contain a simple (respectively, strong) dynamo, because W would never turn black.

An initial set S of black vertices is an irreversible dynamo under simple (respectively, strong) majority if an all black configuration is reached from S in a finite number of steps under the irreversible simple (respectively, strong) majority rule. For brevity we call S a *simple* (respectively, *strong*) *irreversible dynamo*. Similarly a *simple* (respectively, *strong*) *monotone dynamo* is an initial set of black vertices from which an all black configuration is reached in a finite number of steps under the reversible simple (respectively, strong) majority rule, and such that no black vertex ever becomes white during the process. Thus in a monotone dynamo it must never happen that a black vertex has a majority of white neighbours.

Irreversible and monotone dynamos will be treated separately, because they exhibit different properties. Once a majority rule is chosen and consistently applied starting from an initial black set S, each set S' of black vertices obtained in a finite number of steps is said to *derive* from S without repeating the rule applied (with the terminology of cellular automata S' is "in the orbit" of S).

3 Irreversible Dynamos with Simple Majority

Irreversible dynamos generate simpler network evolution than reversible ones. Still, the network behaviour changes drastically if we pass from simple to strong majority. We start our study from the former case.

3.1 Toroidal Mesh

Consider a toroidal mesh with $m \times n$ vertices, $m, n > 2$. Let $S \subseteq T$ be an arbitrary subset of vertices, R_S be the *smallest rectangle* containing S and $m_S \times n_S$ be the size of R_S. If S is all black, a *spanning set* for S (if any) is a *connected* black set $\sigma(S) \supseteq S$ derivable from S. We have:

Lemma 1
Let S be a black set with $m_S < m - 1$ and/or $n_S < n - 1$. Then any (not necessarily connected) black set B derivable from S satisfies $m_B \leq m_S$ and/or $n_B \leq n_S$.

Proof If $m_S < m - 1$, there are $m - m_S \geq 2$ adjacent white rows forming a simple white block that will never turn black. The reasoning for $n_S < n - 1$ is similar. □

Note that this lemma holds also in the case of monotone dynamos.

Theorem 1
Let S be a black set. The existence of a spanning set $\sigma(S)$ implies $|S| \geq \lceil \frac{m_S + n_S}{2} \rceil$.

Proof Let S be a set of minimal cardinality such that $\sigma(S)$ exists. (In fact, if a lower bound on $|S|$ holds for a minimal set, then it must a fortiori hold for any set with the same values of m_S and n_S). We procede by induction on the size of R_S.
Basis We include four cases:

$$m_S = n_S = 1 \qquad \bullet$$

$$m_S = 2,\ \ n_S = 1 \qquad \begin{array}{c}\bullet\\ \bullet\end{array}$$

$$m_S = 1,\ \ n_S = 2 \qquad \bullet\ \ \bullet$$

$$m_S = n_S = 2 \qquad \begin{array}{cc}\bullet & \\ & \bullet\end{array}$$

The hypothesis on $|S|$ is verified. Note that in the first three cases $S = \sigma(S)$.
Induction step. $m_S > 2$ and/or $n_S > 2$. First observe that a minimal set S is not connected. In fact, a connected set with $m_S > 2$ and/or $n_S > 2$ should contain a chain of three adjacent black vertices xyz, where y could be eliminated from S (and then

reconstructed from xz in $\sigma(S)$) without preventing the construction of $\sigma(S)$. Consider now any evolution where $\sigma(S)$ derives from S. Without loss of generality consider to color black one vertex at a time. Starting from (the non connected) S, consider the first step in which the blackening of a white vertex z joins all the vertices of S into a single connected black set $\alpha(S)$, with $\alpha(S) \subseteq \sigma(S)$. Having four neighbours, z joins two, or three, or four disjoint black sets. We then pose $\alpha(S) = \overline{\sigma}(P) \cup \overline{\sigma}(Q) \cup \overline{\sigma}(X) \cup \overline{\sigma}(Y) \cup \{z\}$, where P, Q, X, Y are disjoint subsets of S, with $P \cup Q \cup X \cup Y = S$; P and Q are nonempty; X and/or Y may be empty; $\overline{\sigma}(A)$ denotes a spanning set obtained just before the blackening of z; and $\overline{\sigma}(P), \overline{\sigma}(Q), \overline{\sigma}(X), \overline{\sigma}(Y)$ are also disjoint. We consider three cases:

Case 1. X and Y are empty; there are two black vertices $p \in \overline{\sigma}(P)$ and $q \in \overline{\sigma}(Q)$ adjacent to z; and p, z, q are on the same (horizontal or vertical) straight-line segment. Recall that $\overline{\sigma}(P) \subseteq R_P$, $\overline{\sigma}(Q) \subseteq R_Q$. We have: $m_P + m_Q \geq m_S + 1$, $n_P + n_Q \geq n_S - 1$, if p, z, q are on the same horizontal segment; and $m_P + m_Q \geq m_S - 1$, $n_P + n_Q \geq n_S + 1$, if p, z, q are on the same vertical segment. Hence, in both cases:

$$m_P + n_P + m_Q + n_Q \geq m_S + n_S. \tag{1}$$

By the inductive hypothesis we also have:

$$|P| \geq \lceil \frac{m_P + n_P}{2} \rceil, |Q| \geq \lceil \frac{(m_Q + n_Q)}{2} \rceil. \tag{2}$$

Combining relations 1, 2, and noting that $|S| = |P| + |Q|$, we immediately obtain:

$$|S| \geq \lceil \frac{m_P + n_P}{2} \rceil + \lceil \frac{m_Q + n_Q}{2} \rceil \geq \lceil \frac{(m_S + n_S)}{2} \rceil. \tag{3}$$

Case 2. As in case 1, X and Y are empty, and z is adjacent to two black vertices $p \in \overline{\sigma}(P)$, $q \in \overline{\sigma}(Q)$, but the chain p, z, q is bent. We now have: $m_P + m_Q \geq m_S$ and $n_P + n_Q \geq n_S$, from which we again derive relation 1. Relations 2 and 3 again follow.

Case 3. X and/or Y are not empty. We have: $m_P + m_Q + m_X + m_Y \geq m_S$ and $n_P + n_Q + n_X + n_Y \geq n_S$. Since $|S| = |P| + |Q| + |X| + |Y|$ we easily obtain by induction:

$$|S| \geq \lceil \tfrac{m_P + n_P}{2} \rceil + \lceil \tfrac{m_Q + n_Q}{2} \rceil + \lceil \tfrac{m_X + n_X}{2} \rceil + \lceil \tfrac{m_Y + n_Y}{2} \rceil \geq \lceil \tfrac{(m_S + n_S)}{2} \rceil. \qquad \square$$

An example of set S, and some phases of the black propagation, are illustrated in figure 1.

Combining the results of lemma 1 and theorem 1 we have that a simple irreversible dynamo S must satisfy $m_S > m - 1$ and/or $n_S > n - 1$, hence $|S| \geq \lceil \frac{m-1+n-1}{2} \rceil = \lceil \frac{m+n}{2} \rceil - 1$. This is the first lower bound reported in Table 1.

Let us now define:

An *alternating chain* C as a sequence of adjacent vertices starting and ending with black. The vertices of C are alternating black and white, however, if C has even length there is exactly one pair of consecutive black vertices somewhere. We have:

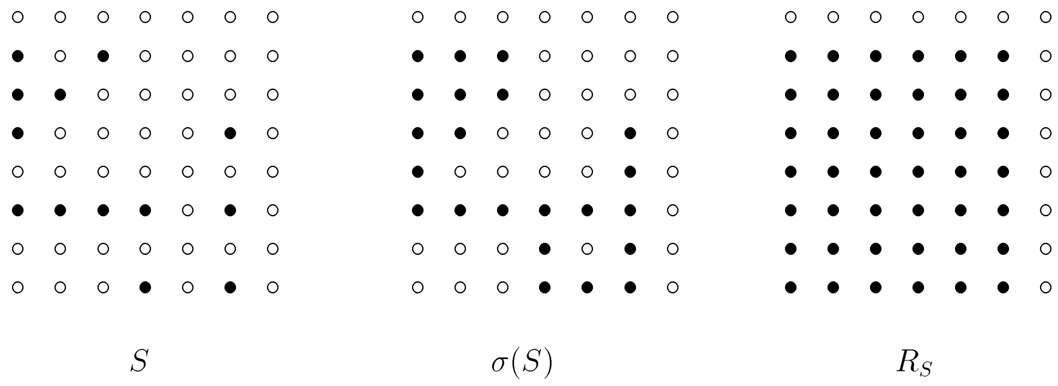


Figure 1: An initial set S; a spanning set $\sigma(S)$; and the smallest rectangle R_S containing S.

Theorem 2
Let C be an alternating chain in a toroidal mesh $m \times n$, placed in column 1, rows 2 to m, and in row m, columns 1 to $n-1$. Then the set S of the black vertices of C is a simple irreversible dynamo with $|S| = \lceil \frac{m+n}{2} \rceil - 1$.

Proof By construction:

1. Color black the whole chain C, thus obtaining a spanning set $\sigma(S)$.

2. Color black R_S as indicated in figure 1. The whole torus is now black, except for one row r and one column c.

3. Color black the vertices of r and c that are adjacent to two vertices of R_S. In fact all the vertices of r and c are in this condition, except for the one at their crossing.

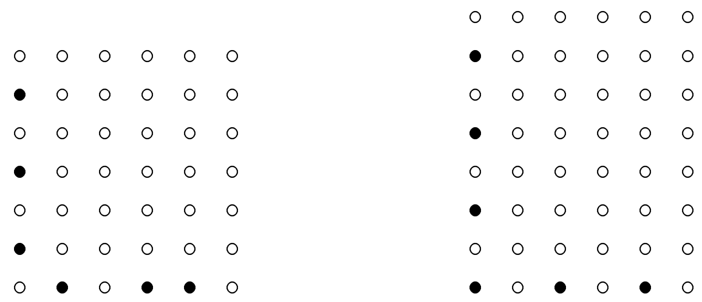
4. Color black the last vertex.

Note now that $|C| = m + n - 3$. Refer to figure 2. If m is even and n is even, or m is odd and n is odd, we have that $|C|$ is odd, hence S contains $\lceil \frac{|C|}{2} \rceil = \lceil \frac{m+n}{2} \rceil - 1$ vertices. If m is even and n is odd, or vice-versa, we have that $|C|$ is even, hence $|S|$ contains $\frac{|C|}{2} + 1 = \lceil \frac{m+n}{2} \rceil - 1$ vertices. $\square$

Note that the lower bound found before matches the upper bound of theorem 2.

3.2 Torus Cordalis and Torus Serpentinus

Although these two families of tori have different properties, the results on simple irreversible dynamos are quite similar.



$m = 7,\ n = 6,\ |S| = 6$ $\qquad$ $m = 8,\ n = 6,\ |S| = 6$

Figure 2: Simple irreversible dynamos in a toroidal mesh.

Theorem 3
Let S be a simple irreversible dynamo for a torus cordalis $m \times n$. Then $|S| \geq \lceil \frac{n}{2} \rceil$.

Proof Immediate from the observation that, in a torus cordalis with simple majority, two consecutive white columns form a simple white block (recall that the first and the last columns are consecutive). □

The lower bound for a torus serpentinus is similar to the one for cordalis, but is a little trickier to prove. Consider a *white cross* C, that is a set of white vertices arranged as in Figure 3, with height m and width n. The vertex at the center of the square of nine vertices inside C is the *center* of the cross. C is a simple white block for a torus serpentinus. Let $N = min\{m, n\}$. We have:

Theorem 4
Let S be a simple irreversible dynamo for a torus serpentinus $m \times n$. Then $|S| \geq \lceil \frac{N}{2} \rceil$.

Proof Starting from a white torus, insert a minimum number of black vertices to eliminate all possible white crosses. Let v be the first black vertex inserted, and consider the cross C centered in v. Exactly all the crosses centered in a vertex of C (including C itself) contain v, hence all these crosses are no more white. Repeat the process blackening new vertices, until no possible white cross remains. Since each new black verex w eliminates exactly all the crosses centered in a vertex of the cross D centered in w, and D invades two rows and two columns of each side of w, we must include at least $\lceil \frac{N}{2} \rceil$ black vertices to be sure that no white cross remains. □

Theorem 5
In a torus cordalis $m \times n$, the set S of $\lfloor \frac{n}{2} \rfloor + 1$ black vertices shown in figure 4 is a simple irreversible dynamo.

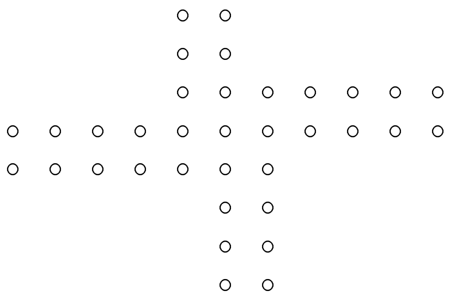


Figure 3: The white cross configuration in a torus serpentinus.

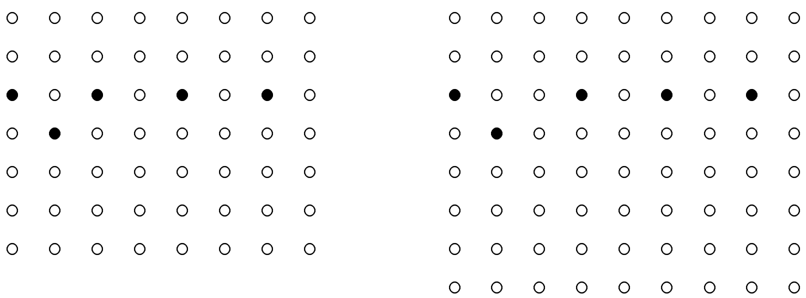


Figure 4: Simple irreversible dynamos of $\lfloor \frac{n}{2} \rfloor + 1$ vertices for torus cordalis and serpentinus, with n even and n odd. Symmetric configurations, exchanging rows and columns, are simple irreversible dynamos for torus serpentinus. In this case we can choose a dynamo of $\lfloor \frac{N}{2} \rfloor + 1$ vertices.

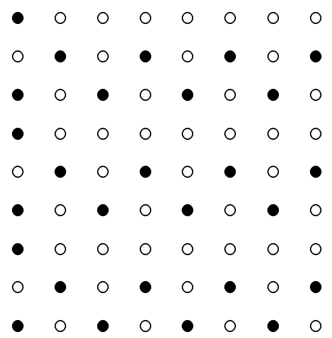
Theorem 6
In a torus serpentinus $m \times n$, the set S of $\lfloor \frac{N}{2} \rfloor + 1$ black vertices presented in figure 4 is a simple irreversible dynamo.

4 Irreversible Dynamos with Strong Majority

For strong majority we can derive a significant lower bound valid for the three considered families of tori (simply indicated as tori). Since these tori have a neighbourhood of four, three adjacent vertices are needed to color black a white vertex under strong majority. We have:

Theorem 7
Let S be a strong irreversible dynamo for a torus $m \times n$. Then $|S| \geq \lceil \frac{mn+1}{3} \rceil$.

Proof Let T and E be the sets of vertices and edges of the torus; $S \subseteq T$ be the set of black vertices; R be the restriction of the torus to the subset $T - S$ of white vertices. If S is a dynamo, R cannot contain any cycle that would be a strong white block. Then R must be a forest, and its set E_R of edges is such that $|E_R| \leq |T| - |S| - 1$. Note that each edge of the set $E - E_R$ has at least one extreme in S. Since each vertex of T has four neighbours, we have $|E| - |E_R| \leq 4|S|$. We then have: $|E| = 2mn \leq |T| - |S| - 1 + 4|S| = mn + 3|S| - 1$ and the bound follows. □



$m = 9,\ n = 8,\ |S| = \lceil \frac{m}{3} \rceil (n+1) = 27$

Figure 5: A strong irreversible dynamo for toroidal mesh, torus cordalis or torus serpentinus.

We now derive an upper bound also valid for all tori.

Theorem 8
Any torus $m \times n$ admits a strong irreversible dynamo S such that $|S| = \lceil \frac{m}{3} \rceil (n+1)$.

Proof Let each group of three consecutive rows be colored as follows: All the vertices in the first row are white, except for the first one; the vertices in the second and in the third row have alternating colors, respectively starting with white and with black (see figure 5). If m is not a multiple of three, make the same configuration without the first row, or the first and the second row. We have $\lceil \frac{m}{3} \rceil (n+1)$ black vertices. It can be easily seen that, after one step, the second and third rows in each group of three, and the first column, become black. At each consecutive step two new columns, adjacent to the columns already colored black, become also black, until the whole torus is colored black. □

For particular values of m and n the bound of theorem 8 can be made stricter for the toroidal mesh and the torus serpentinus. In fact these networks are symmetrical with respect to rows and columns, hence the pattern of black vertices reported in figure 5 can be turned of 90 degrees, still constituting a dynamo. We immediately have:

Corollary 1
Any toroidal mesh or torus serpentinus $m \times n$ admits a strong irreversible dynamo S with $|S| = max\{\lceil \frac{m}{3} \rceil (n+1), \lceil \frac{n}{3} \rceil (m+1)\}$.

5 Monotone Dynamos with Simple Majority

As for the irreversible case, the network behaviour changes drastically depending on the type of majority applied. In this section we apply reversible simple majority. To attain monotonicity, we must be sure that a black vertex is never left with more than two white neighbours. For this we introduce a new configuration.

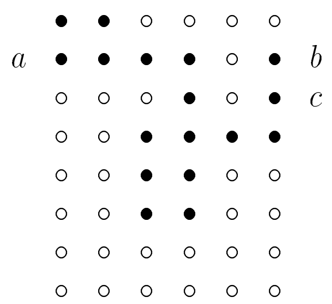


Figure 6: A black compact in a toroidal mesh: note that vertex b is adjacent to two black vertices a, c.

A *black compact* C is a connected subset of black vertices, each of which has at least two neighbours in C. Notice that, under both simple and strong majority rules, the vertices of a black compact will never become white. An example of black compact is shown in Figure 6. Other examples are a black column in a toroidal mesh or in a torus cordalis; a black row, or a black column in a toroidal mesh etc.

By the definitions of black compact and white block, we have for all tori:

Lemma 2
Let S be a simple (respectively strong) monotone dynamo of any torus. Then:
(i) S is a collection of black compacts;
(ii) $T - S$ does not contain any simple (respectively strong) white block.

5.1 Toroidal Mesh

Consider a toroidal mesh $m \times n$ with $m, n > 2$ vertices, and let $S \subseteq T$. First note that lemma 1 holds also in this case. A *black gallows* G as any subset of T composed of all black vertices lying in a whole row and in a whole column of the mesh. A black gallows G is also a black compact, and we have $|G| = m + n - 1$.

To establish a lower bound to the cardinality of a monotone dynamo, we first prove an elementary property of black compacts.

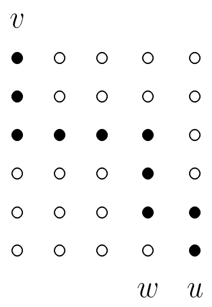


Figure 7: A portion $v - u$ of the spanning path of lemma 3 in the submesh $m_C \times n_C$.

Lemma 3
Let C be a black compact of a toroidal mesh $m \times n$. Then:
(i) if $m_C = m$ and/or $n_C = n$ then $|C| \geq m_C + n_C - 1$;
(ii) if $m_C \leq m - 1$ and $n_C \leq n - 1$ then $|C| \geq m_C + n_C$.

Proof Traverse C starting from one of its vertices v, moving along vertical and horizontal mesh edges spanning over m_c different rows and n_c different columns. We need at least $m_C - 1$ vertical steps and $n_C - 1$ horizontal steps, thus touching at least $m_C + n_C - 1$ different vertices including v. This is enough to prove point (i). (Indeed the bound $m_C + n_C - 1$ is met only for $m_C = m$ and $n_C = n$, where the smallest compact C is a black gallows; or for $m_C = m$ and $n_C = 1$ where C is a column; or for $m_C = 1$ and $n_C = n$ where C is a row. In all the other cases we have $|C| > m_C + n_C - 1$).
To prove point (ii), note that the spanning path from v cannot terminate in a vertex u with only one black neighbour, because C is a compact. To prevent this situation at least another neighbour w of u must be present in C, as indicated in figure 7. This is enough to prove point (ii). (Indeed the bound $m_C + n_C$ is met only for $m_C = n_C = 2$, where C is a square of four vertices and w is an additional neighbour both for u and v. Otherwise also v, and possibly other vertices of C not shown in the figure, must be provided with an additional black neighbour, thus increasing the bound over $m_C + n_C$).
□

We can now state:

Theorem 9
Let S be a monotone irreversible dynamo for a toroidal mesh of size $m \times n$. We have:
(i) $m_S \geq m - 1, \quad n_S \geq n - 1$;
(ii) $|S| \geq m + n - 2$.

Proof Point (i) is immediately proved as a consequence of lemma 1, noting that $T-S$ cannot have two consecutive white rows or columns which constitute a simple white block. To prove point (ii), let $S = C_1 \cup C_2 \cup \ldots \cup C_k$ be a collection of black compacts, $k \geq 1$ (lemma 2). We consider two cases.
Case 1. One of the black compacts, say C_1, has $m_{C_1} = m$ and/or $n_{C_1} = n$. If $m_{C_1} + n_{C_1} \geq m+n-1$ we have $|C_1| \geq m+n-2$ by lemma 3 (i), and the theorem is proved. Now let $m_{C_1} = m$, $n_{C_1} < n-1$ (the case $m_{C_1} < m-1$, $n_{C_1} = n$ is symmetric). Other black compacts $C_2, \ldots, C_k$ must provide black vertices in some of the $n - n_{C_1}$ columns outside R_{C_1}, so that $n_S \geq n-1$ (by the already proved point (i) of this theorem), and no two adjacent such columns are white. It can be immediately verified that, in a set of minimal cardinality with this property, each C_i must have $m_{C_i} < m$, $2 \leq i \leq k$. In fact, each C_i has $nC_i < n-1$, otherwise C_i would intersect C_1; and $m_{C_i} \geq 2$, otherwise C_i would not be a compact. By lemma 3 (ii) we have: $|C_2| + \ldots + |C_k| \geq m_{C_2} + \ldots + m_{C_k} + n_{C_2} + \ldots + n_{C_k} \geq 2(k-1) + n_{C_2} + \ldots + n_{C_k}$, and we must minimize this function with the condition $n_{C_2} + \ldots + n_{C_k} \geq n - n_{C_1} - k$, where the term $-k$ denotes the existence of at most one white column between any pair of black compacts. We then have: $|C_2| + \ldots + |C_k| \geq n - n_{C_1} + k - 2$, that is minimized for $k = 2$ (i.e., only two black compacts C_1, C_2 are present). In conclusion we have: $|S| = |C_1| + |C_2| \geq m + n_{C_1} - 1 + n - n_{C_1} = m+n-1$.
Case 2. All the black compacts C_i have $m_{C_i} < m$ and $n_{C_i} < n$. If one such compact, say C_1, has $m_{C_1} = m-1$ and $n_{C_1} = n-1$, then $|C_1| \geq m+n-2$ by lemma 3 (ii), and the theorem is proved. Otherwise S must contain more than one such compact. The proof is by induction on the size of the sets built from S by successive applications of the reversible majority rule. Consider the first step in which a unique connected set $\alpha(S) \supset S$ is generated, with a white vertex z turning black to join several disjoint black compacts. It can be easily verified that the possible black neighbours of any vertex cannot belong to more than two black compacts. z joins exactly two disjoint black subsets, say $\bar{\sigma}(P)$, $\bar{\sigma}(Q)$, with $P = C_1 \cup \ldots \cup C_h$, $Q = C_{h+1} \cup \ldots \cup C_k$. We have: $m_P + n_P + m_Q + n_Q \geq m_S + n_S$. Inductively assume: $|P| \geq m_P + n_P$, $|Q| \geq m_Q + n_Q$, where the basis of the induction for a single C_i is given by lemma 3 (ii). From the two relations above we have: $|S| = |P| + |Q| \geq m_S + n_S$, and from point (i) of this theorem we finally derive: $|S| \geq m+n-2$. □

We now build a monotone dynamo whose size is close to the bound of theorem 9.

Theorem 10
In a toroidal mesh $m \times n$, a black gallows G is a simple monotone dynamo of cardinality $m+n-1$.

Proof Without loss of generality let G lay on row 0 and column 0. Color black the vertices in row 1, columns 1 to $n-1$, then in row 2 etc. Note that each black configuration thus derived is a black compact. □

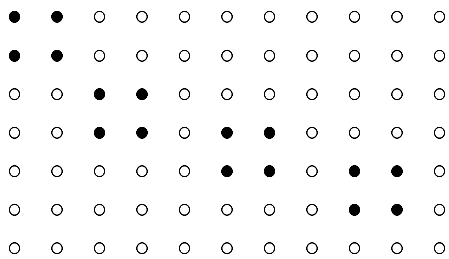


Figure 8: A collection C_0, C_1, C_2, C_3 of black compacts in a 7×11 toroidal mesh. $C = C_0 \cup C_1 \cup C_2 \cup C_3$ is a monotone dynamo with $|C| = m + n - 2$.

The upper bound of Theorem 10 differs only by 1 from the lower bound of Theorem 9. However, an upper bound of $m+n-2$ can be derived for an infinite subfamily of toroidal meshes with special dimensions m, n. Namely, let: $m + n = 6 + 4k$ ($k = 0, 1, \ldots$), and $|m - n| = 2r$ ($0 \leq r \leq k$). Proper values are, for example, $m = 42$ and $n = 60$, for which we have $m + n = 6 + 4 \cdot 24$, $|m - n| = 2 \cdot 9$. Build a collection of black compacts $C_0, C_1, \ldots, C_k$, each composed of four black vertices arranged in a square of side 2. For $0 \leq i \leq r$, place each C_i in rows $2i$, $2i + 1$, and in columns $2i$, $2i + 1$. For $m > n$ (hence $r < k$), and for $1 \leq j \leq k - r$, place C_{r+j} in rows $2r + 3j$, $2r + 1 + 3j$, and in columns $2r + j$, $2r + 1 + j$. For $n > m$ (hence $r < k$), and for $1 \leq j \leq k - r$, place C_{r+j} in rows $2r + j$, $2r + 1 + j$, and in columns $2r + 3j$, $2r + 1 + 3j$. A collection of black compacts $C_0, \ldots, C_3$ for $m = 7$, $n = 11$, is shown in figure 8, where we have $m + n = 18$, $k = 3$, $r = 2$, $n > m$. The reader may immediately discover that $C = \bigcup_{i=0}^{k} C_i$ is a monotone dynamo, with $|C| = 4 \cdot (k + 1) = m + n - 2$.

5.2 Torus Cordalis and Torus Serpentinus

For the tori cordalis and serpentinus we can establish lower and upper bounds that match exactly. We have:

Theorem 11
Let S be a monotone dynamo for a torus cordalis $m \times n$. Then $|S| \geq n + 1$.

Proof From lemma 2, let $S = C_1 + \ldots + C_k$ be a collection of black compacts. If one such a compact C_i has $n_{C_i} = n$, we have $|S| \geq |C_i| \geq n+1$ because a single black row, or a "snake-like" black row containing the vertices $\tau_{i,j} \ldots \tau_{i,n-1} \tau_{(i+1) \bmod m, 0} \ldots \tau_{(i+1) \bmod m, j-1}$, is not a compact in this torus, unless another black vertex is added to close a cycle. If $n_{C_i} < n$, for all i, we must have $n_{C_1} + \ldots + n_{C_k} \geq n - k$, to prevent the existence of a simple white block composed of two adjacent white columns. For each C_i to be a compact we must have $|C_i| \geq n_{C_i} + 2$ (as in the proof of lemma 3). Then we have: $|C| = |C_1| + \ldots + |C_k| \geq n_{C_1} + \ldots + n_{C_k} + 2k \geq n + k$. □

The matching upper bound is the following, by immediate inspection:

Theorem 12
In a torus cordalis of size $m \times n$, the set S of $n+1$ black vertices placed in a whole row i, and in row $(i+1) mod\, m$, column 0, is a simple monotone dynamo.

Let $N = min\{m, n\}$. In a similar way we can prove that:

Theorem 13
Let S be a monotone dynamo for a torus serpentinus $m \times n$. Then: $|S| \geq N + 1$.

Theorem 14
In a torus serpentinus $m \times n$, the set S of $N+1$ black vertices placed in a whole row i, and in row $(i+1) mod\, m$, column 0, for $N = n$; or placed in a whole column j, and in column $(j+1) mod\, n$, row 0, for $N = m$; is a simple monotone dynamo.

6 Monotone Dynamos with Strong Majority

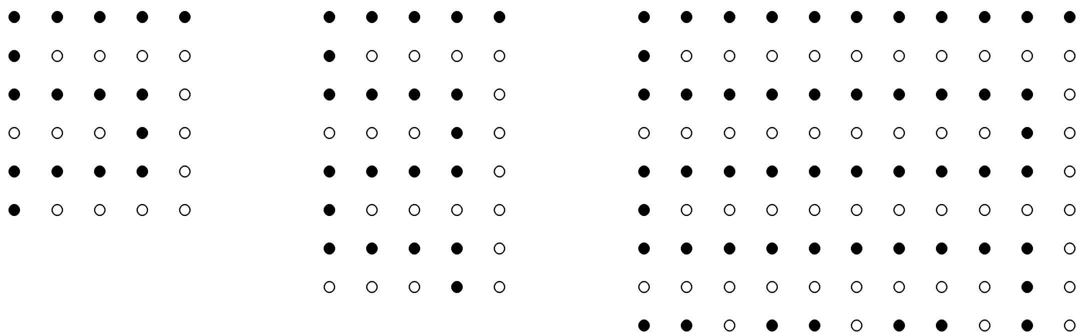
This is the simplest case. To rule out the presence of strong white blocks we must impose that the restriction of the torus to the subset of initial white vertices is a forest. Moreover, in order to ensure monotonicity, the initial black vertices must form a collection of black compacts.

A single argument yields a significant lower bound for the three families of tori. We have:

Theorem 15
Let S be a strong monotone dynamo for a torus $m \times n$. Then $|S| \geq \lceil \frac{mn+1}{2} \rceil$.

Proof Let T and E be the sets of vertices and edges of the torus. S and $T - S$ are the subsets of initial black and white vertices, respectively. Let E_{WW} and E_{WB} be the subset of edges connecting two white vertices and a white vertes with a black vertex respectively. Since each white vertex has four incident edges, and the edges of E_{WW} connect two white vertices, we immediately have: $|E_{WB}| = 4|T - S| - 2|E_{WW}|$. Now the restriction of the torus to the subset of white vertices must be a forest, hence we have: $|E_{WW}| \leq |T - S| - 1$, hence $|E_{WB}| \geq 4|T - S| - 2(|T - S| - 1) = 2|T| - 2|S| + 2$. Since S must be a collection of compacts, each of its vertices can have at most two leaving edges that belong to E_{WB}, hence $|E_{WB}| \leq 2|S|$. Combining the two inequalities for $|E_{WB}|$ we obtain $2|S| \geq |T| + 1$ that proves the theorem. $\Box$

We now derive an upper bound valid for all three families of tori. Let $N = min\{m, n\}$. We have:



m and/or n even $\qquad\qquad$ m and n odd

Figure 9: Strong monotone dynamos for the three families of tori.

Theorem 16
Any torus $m \times n$ admits a strong monotone dynamo S, with $|S| = \frac{mn}{2} + 1$ for m and/or n even, or $|S| = \lceil \frac{mn}{2} + \frac{N}{6} + \frac{2}{3} \rceil$ for m and n odd.

Proof Assign the vertices of S as shown in Figure 9. Note that the black vertices form a compact and the white vertices form a tree. The theorem easily follows from inspection of the figure and all its significant symmetries. □

7 Concluding Remarks

In this paper, we have established upper and lower bounds on the size of irreversible and monotone dynamos in tori for different majority rules, summarized in table 1. In both monotone and irreversible dynamos the bounds are tight within an additive constant, thus providing an accurate quantitative description of the size of dynamos in tori.These bounds also constitute the first precise look on the relationship between irreversible and reversible monotone dynamos.

As indicated in the table, in the case of *simple majority*, the bounds for irreversible dynamos are smaller by a factor of two than the ones for reversible monotone dynamos. In the case of *strong majority* the gap is narrower: the constant becomes 3/2.

This raises the intriguing question of whether it is possible to transform an irreversible dynamo into a monotone one using at most twice the number of initial black nodes (at most 3/2 in the case of strong majority). Should this not be the case, the next question would be whether it can be done for some constant k.

References

[1] Z. Agur, A. S. Fraenkel, S. T. Klein. The number of fixed points of the majority rule. *Discrete Mathematics*, 70:295–302, 1988.

[2] J-C Bermond, J. Bond, D. Peleg, S. Perennes. Tight bounds on the size of 2-monopolies *Proc. of 3rd Colloquium on Structural Information and Communication Complexity* (SIROCCO 96), Siena, 170-179, 1996.

[3] J.-C. Bermond, F. Comellas, D.F. Hsu. Distributed loop computer networks: a survey. *Journal of Parallel and Distributed Computing*, 24:2-10, 1995

[4] J-C Bermond, D. Peleg. The power of small coalitions in graphs. In *Proc. of 2nd Colloquium on Structural Information and Communication Complexity* (SIROCCO 95), Olympia, 173-184, 1995.

[5] R. De Prisco, A. Monti, L. Pagli. Efficient testing and reconfiguration of VLSI linear arrays. *Theoretical Computer Science*, vol. 197, 171-188, 1998.

[6] P. Flocchini, F. Geurts, N. Santoro. Irreversible dynamos in chordal rings. *Discrete Applied Mathematics*, to appear. Preliminary version in *Proc. 25th International Workshop on Graph-Theoretic Concepts in Computer Science* (WG 99), 1999.

[7] P. Flocchini, E. Lodi, F. Luccio, L. Pagli, N. Santoro. Irreversible dynamos in tori. *Proc. European Conference on Parallel Computing* (EUROPAR 98), Southampton, 554-562,1998.

[8] P. Flocchini, E. Lodi, F. Luccio, L. Pagli, N. Santoro. Monotone dynamos in tori. *Proc. of 6th Colloquium on Structural Information and Communication Complexity* (SIROCCO 99), 152-165, 1999.

[9] P. Flocchini, R. Kralovic, A. Roncato, P. Ruzicka, N. Santoro. On time versus size for monotone dynamic monopolies in regular topologies. *Journal of Discrete Algorithms*, to appear. Preliminary version in *Proc. 7th International Colloquium on Structural Information and Communication Complexity* (SIROCCO 2000), 111-126, 2000.

[10] E. Goles, J. Olivos. Periodic behavior of generalized threshold functions. *Discrete Mathematics*, 30:187–189, 1980.

[11] A. Granville. On a paper by Agur, Fraenkel and Klein. *Discrete Mathematics*, 94:147–151, 1991.

[12] N. Linial, D. Peleg, Y. Rabinovich, M. Sachs. Sphere packing and local majority in graphs In *Proc. of 2nd ISTCS*, IEEE Comp. Soc. Press, 141-149, 1993.

[13] F. Luccio, L. Pagli, H.Sanossian. Irreversible dynamos in butterflies. *Proc. of 6th Colloquium on Structural Information and Communication Complexity* (SIROCCO 99), 204-218, 1999.

[14] G. Moran. Parametrization for stationary patterns of the r-majority operators on 0–1 sequences. *Discrete Mathematics*, 132:175–195, 1994.

[15] G. Moran. The r-majority vote action on 0–1 sequences. *Discrete Mathematics*, 132:145–174, 1994.

[16] G. Moran. On the period-two-property of the majority operator in infinite graphs. *Transactions of the American Mathematical Society*, 347(5):1649–1667, 1995.

[17] A. Nayak, N. Santoro, R. Tan. Fault intolerance of reconfigurable systolic arrays. *Proc. 20th International Symposium on Fault-Tolerant Computing*, 202-209, 1990.

[18] D. Peleg. Local majority voting, small coalitions and controlling monopolies in graphs: A review. *Proc. of 3rd Colloquium on Structural Information and Communication Complexity* (SIROCCO 96), Siena, 152-169, 1996.

[19] D. Peleg. Size bounds for dynamic monopolies *Proc. of 4th Colloquium on Structural Information and Communication Complexity*(SIROCCO 97), Ascona, 151-161, 1997.

[20] S. Poljak. Transformations on graphs and convexity. *Complex Systems*, 1:1021–1033, 1987.

[21] S. Poljak, M. Sůra. On periodical behavior in societies with symmetric influences. *Combinatorica*, 1:119–121, 1983.

[22] N. Santoro, J. Ren, A. Nayak. On the complexity of testing for catastrophic faults. *Proc. of 6th International Symposium on Algorithms and Computation* (ISAAC 95), 188-197, 1995.

[23] N. Zeh, N. Santoro. On finding minimum deadly sets for directed networks. *Proc. of 8th Colloquium on Structural Information and Communication Complexity* (SIROCCO 01), 351-365, 2001.