Optimal Irreversible Dynamos in Chordal Rings

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Abstract

In this paper we study the effect of a simple majority rule on two classes of chordal rings: weakly chorded and strongly chorded rings. In the case of weakly chorded rings, we establish a lower bound on the weight of optimal dynamos and we prove that the bound is tight with a constructive upper bound; we also provide a complete characterization of the optimal dynamos for the well known class of double and triple loop networks. Also in the case of strongly chorded rings, we establish tight bounds and show how to construct optimal dynamos.

Key words: Majority Rule, Dynamic Monopolies, Chordal Rings.

1 Introduction

Let G be a simple connected graph where every node is colored either *black* or *white*. Consider now the following repetitive process on G: each node recolors itself, at each local time step, with the color held by the majority of its neighbors. Since the clocks are not necessarily synchronized, the process can be asynchronous.

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Depending on the initial assignment of colors to the nodes and on the definition of majority, different *dynamics* can occur. These dynamics have been extensively studied in the context of *synchronous* systems (mostly cellular automata); researchers mostly concentrated their efforts on determining *periodic* behaviors, transients, number and form of fixed points for different graph structures (rings, infinite lines, general finite graphs, general infinite graphs) with different majority functions (simple majority, strong majority, weighted threshold functions, convex functions) and different coloring sets (e.g., see [1,11,16,17,22]).

In the context of *distributed computing*, this repetitive process is particularly important in that it describes the impact that a set of initial faults can have in majority-based systems (where black nodes correspond to faulty elements and white to non-faulty ones). Consider for example systems where majority voting among various copies of crucial data are performed between neighbors at each step [20]: if the majority of its neighbors is faulty (e.g., has corrupted data), a non-faulty element will exhibit a faulty behavior (e.g., its data will become corrupted) and will therefore be indistinguishable from a faulty one. In this paper, we are interested in the patterns of initial faults which may lead the entire system to a faulty behavior (e.g., every entity has corrupted data); in terms of system dynamics, these are the patterns for which the system converges to a *monochromatic* fixed point. These patterns are called *dynamos* (short for "dynamic monopolies") and their study has been introduced by Peleg [21]. If the initial faults are permanent, the dynamo is said to be *irreversible*; if instead the initial faults can be mended by the majority rule, the dynamo will be called *reversible*.

Surprisingly, very little is known about dynamos. Most of the results are known for the *static* version of this process; that is, considering only a single step in the evolution [3,5,13,20].

Recently, researchers have started to focus directly on dynamos. In particular, the problem is introduced in [21], where some generic observations on reversible monotone dynamos are provided; monotone and irreversible dynamos for tori have been investigated in [7,8], where lower and upper bounds on the size of optimal dynamos are given; bounds for optimal dynamos in butterfly and similar interconnection networks have been shown in [14]; finally, the issue of time to converge to a monochromatic configuration has been raised in [9], where trade-offs between size of a dynamo and convergence time are studied in several topologies.

In this paper we study irreversible dynamos for (sufficiently large) finite *chordal rings*, i.e. ring networks where each node is also directly connected to the nodes at a fixed set of distances, under the *simple majority* rule: a node becomes black if a simple majority of its neighbors are black. More specifically, we concentrate on two large classes of chordal rings, namely weakly and strongly chorded rings. Given a dynamo for these graphs, its *weight* is the number of black nodes it contains, and its *length* is the size of the (smallest) segment of the ring containing all the back elements. We are interested in characterizing *optimal dynamos*, that is dynamos of minimum length with the minimal weight.

Chordal rings are a particular case of *circulant graphs*, and are also known in the literature as *distributed loop networks*. Weakly and strongly chorded rings include many of the most studied chordal rings, ranging from double loops to fan graphs to complete graphs. Special classes of chordal rings (in particular the so called double loops and triple loops) have been widely studied to analyze their fault tolerant properties (for a survey see [4]). Subclasses of chordal rings have been intensively studied under a variety of scenarios and for a large number of problems and applications ranging from routing (e.g., [12]) to election (e.g., [2,15,19]) to broadcast (e.g., [15]). Of particular relevance are also the existing studies on *catastrophic fault patterns* for redundant linear arrays (e.g., see [6,18,23]); in fact, these patterns are exactly irreversible dynamos for infinite chordal rings under a *directional unanimity* rule: a node becomes black if all its neighbors in the same direction are black.

In the following sections of this paper, we first establish some basic bounds on the weight and length of dynamos for general chordal rings (§2). We then consider weakly chorded rings (§3). We establish a *lower bound* on the weight of dynamos of minimal length, and prove that the bound is *tight*; the upper bound proof is *constructive*. We also provide a *complete characterization* of the optimal dynamos for the well known class of double loop networks and the subclass of triple loops. In Section §4, we study strongly chorded rings and their optimal dynamos; also for this class of graphs we establish *tight bounds*, and show how to construct optimal dynamos. Finally, in section §5, we make some concluding remarks.

2 Definitions and Basic Bounds

A *circulant graph* of size n and link structure $\langle d_1, d_2, \cdots, d_h\rangle$ ($d_i \leq d_{i+1}$, $d_h \leq \lfloor \frac{n}{2} \rfloor$), is a graph on n nodes $x_0, x_1, \cdots, x_{n-1}$ where each node x_i is connected to the nodes x_{i+d_j} and x_{i-d_j} ($1 \leq j \leq h$). We will often use "x_i *sees* x_j" to mean that node x_i and node x_j are connected to each other and all operations on the indices are modulo n. We shall denote such a graph by $C\{d_1, d_2, \cdots, d_h\}$. A *chordal ring* (or *loop network*) is any circulant graph with $d_1 = 1$.

A (p, k)-*chorded ring*, $p \leq k$, is the circulant graph $C\{1, 2, \cdots, p, k\}$. We shall denote such a graph by $C\langle p, k\rangle$. Examples of (p, k)-chorded rings are rings

($p = k = 1$), double loop networks ($p = 1, k > 1$), fan networks ($p = k - 1$), and complete graphs ($p = k = \lfloor \frac{n}{2} \rfloor$). Depending on the relationship between p and k a chorded ring will be said to be *weakly* ($p < \lfloor \frac{k}{2} \rfloor$) or *strongly* ($p \geq \lfloor \frac{k}{2} \rfloor$).

Let G be the chordal ring $C\{d_1, d_2, \cdots, d_h\}$. In the following, and unless otherwise stated, we will assume that all d_i are distinct.

The nodes of G performs the following repetitive process. Initially, each node is in one of two states: *black* or *white* (denoted in also by "$\mathbf{(0)}$" and "$\mathbf{(1)}$", respectively). An initially black node does not change its value (*irreversibility*). Each other node at each time step becomes black if and only if the simple majority of its neighbors are black. This simple majority rule corresponds to the "self-not-included, prefer-black" model of [21]. Notice that the assumption of irreversibility of the initial black nodes implies that the evolution of the graph is *monotonic*, i.e., once a node becomes black, it remains black forever.

Since the clocks are not necessarily synchronized, the process can be asynchronous. However, since the evolution of the system is monotonic, we can assume synchronicity without any loss of generality [10,21].

A *pattern* P is a sequence of values of consecutive nodes in the ring at a given global time; its *weight* is the number of black values it contains; its *length* is its total number of nodes. A pattern is *bounded* if its first and last values are both black; it is *converging* if, after a finite number of steps, all its nodes become black.

A *configuration* Y is a global state representing the colors of all the nodes at a global time. The *window* $W(Y)$ of Y is the smallest pattern containing all the black nodes; we denote by $weight(Y)$ and $length(Y)$ respectively the weight and length of its window $W(Y)$.

A *dynamo* is a configuration from which an all black configuration is reached. A dynamo is *optimal* if it has minimum weight and minimal length.

Let us first consider the minimum *weight*; i.e., the minimum number of initial black nodes needed to reach an all-black configuration. An *upper bound* is easily established, observing that any configuration Y containing d_h consecutive blacks is obviously a dynamo for $C\{1, d_2, \cdots, d_h\}$.

Property 1 *Let Y be a configuration in $C\{1, d_2, \cdots, d_h\}$ which contains d_h consecutive blacks (i.e., the pattern $\mathbf{(1)}^{d_h}$); then Y is a dynamo.*

PROOF. By simple induction, observing that any white node adjacent to d_h consecutive black nodes has h black neighbors and hence becomes black by the simple majority rule. □

An obvious lower bound on the weight of any dynamo X in G is $weight(X) \geq h$, since otherwise no node will ever become black. It is interesting to note that there exist infinite families of chordal rings having dynamos achieving such a bound, for example, as we will show later, this holds for strongly chorded rings. Summarizing,

Property 2 *Let X be a minimum weight dynamo for $C\{1, d_2, \cdots, d_h\}$; then $h \leq weight(X) \leq d_h$.*

In the case of chorded rings this becomes:

Corollary 1 *Let X be an optimal dynamo for $C\langle p, k\rangle$, $p < k$; then: $p + 1 \leq weight(X) \leq k$.*

Consider now $length(X)$; i.e., the size of the window containing all the black nodes of a dynamo X. An obvious lower bound follows from the definition: $weight(X) \leq length(X)$ since all black nodes must be contained in the window. A stronger lower bound is stated in the next property.

Property 3 *Every dynamo X for $C\{1, d_2, \cdots, d_h\}$, is such that $d_h \leq length(X)$.*

PROOF. We prove that when $length(X) < d_h$, no node outside the window $W(X)$ can change its value even if all nodes in the windows are black. By contradiction, let $length(X) < d_h$ and let v be a node outside the window that becomes black in the first step (i.e., due solely to the influence of the window). Assume without loss of generality that v is on the right of the window; then, obviously, all its right neighbors are white, and its black neighbors can only be located to its left. If v is adjacent to the window, since $length(X) < d_h$, its left neighbor at distance d_h is not part of the window; thus, v has only $h - 1$ black neighbors, and cannot become black: a contradiction. If v is not adjacent to the window, its left neighbor in the ring is also white because it is not part of the window; it follows that, also in this case, v has at most $h - 1$ black neighbors, and thus cannot become black: a contradiction. $\square$

In the case of chorded rings this becomes:

Corollary 2 *Let X be a dynamo for $C\langle p, k\rangle$, $p < k$; then: $k \leq length(X)$.*

3 Weakly Chorded Rings

A chorded ring $C\langle p,k\rangle$ is said to be *weakly* if $p < \lfloor \frac{k}{2} \rfloor$. In this Section we consider only weakly chorded rings and establish tight bounds on their optimal dynamos. We then study in details two particular cases: the (simple) well known class of double loop networks and the (more complex) subclass of triple loops $C\langle 2,k\rangle$; for both we provide a complete characterization of the optimal dynamos.

3.1 Lower Bound

Any dynamo of length k is length-minimal (see Corollary 2). In this section we establish a lower bound on the weight of such a dynamo.

We first characterize a *forbidden* pattern for dynamos of length k. Let V be the set of nodes of the chordal ring $C\langle p,k\rangle$; a *white block* $B \subseteq V$ is a subset of V composed of all white vertices, each of which has at least $p+2$ neighbors in B; hence, nodes of a white block will never become black. Thus,

Lemma 1 *No dynamo for $C\langle p,k\rangle$ can contain a white block.*

We now characterize several types of white blocks. The first one concerns infinite chordal rings.

Lemma 2 *Let $C\langle p,k\rangle$ be infinite, R_i be the set of consecutive nodes $x_i \ldots x_{i+p}$, and $R[i]$ be the set of nodes $R[i] = \cup_{j=0}^{\infty} R_{i\pm jk}$. If all nodes in $R[i]$ are white, then $R[i]$ is a white block.*

PROOF. Each $R_{i\pm jk}$ forms a white clique of degree p. Moreover, each node x_l in $R[i]$ is linked to the two nodes x_{l+k} and x_{l-k} which both belong to $R[i]$. In other words, every node in $R[i]$ has $p+2$ neighbors in $R[i]$; thus, $R[i]$ is a white block. □

The following lemma gives a type of white blocks for sufficiently large chorded rings.

Lemma 3 *Let $C\langle p,k\rangle$ be finite and of length n. Let $A_i = a_0 \ldots a_p$ and $B_i = b_0 \ldots b_{n-2k+p}$ be the sets of consecutive nodes $x_i \ldots x_{i+p}$ and $x_{i+k} \ldots x_{i+n-k+p}$, and let all nodes in $A_i \cup B_i$ be white. If one of the following conditions holds, then, $A_i \cup B_i$ is a white block:*

(1) $p = 1$ and $n \geq 4k-2$ (i.e., $|B_i| \geq 2k$);

(2) $p > 1$ and $n \geq 3k - p$ (i.e., $|B_i| \geq k + 1$).

PROOF. We have to prove that every node of $A_i \cup B_i$ is connected to at least $p + 2$ other nodes in $A_i \cup B_i$.

Under both conditions, A_i forms a white clique of degree p. Moreover, since for $i \leq p, 1 \leq k - (p - i) - (k - p - 1) \leq |B_i|$, the chord at distance k of an arbitrary node in A_i falls in B_i. Thus, each node in A_i is connected to exactly $p + 2$ nodes in $A_i \cup B_i$.

We now show that an arbitrary node b_j of B_i is connected to $p + 2$ nodes in $A_i \cup B_i$; we consider the case $j \leq \lceil \frac{|B_i|}{2} \rceil - 1$, the other cases follow symmetrically. Under both conditions, we have that $\lceil \frac{|B_i|}{2} \rceil - 1 + p \leq |B_i| - 1$; thus, node b_j is connected to the p nodes $b_{j+1}, \ldots, b_{j+p}$. The additional nodes to which it is connected depend on the condition.
Under condition 1: From $|B_i| \geq 2k$ it follows that $\lceil \frac{|B_i|}{2} \rceil - 1 + k \leq |B_i| - 1$; thus, node b_j is connected to b_{j+k}. Moreover, if $j \leq 1$ node b_j is also connected to a_j (the node at distance k to its left) otherwise it is connected to b_{j-1}.
Under condition 2: If $j = 0$, b_0 is connected to a_0 and b_k (since $k \leq |B_i| - 1$). If $j = 1$, it is connected to a_1 and b_0. Finally, if $j \geq 2$, it is connected to b_{j-1} and b_{j-2} (since $p > 1$).
Therefore, each node in B_i is connected to $p + 2$ nodes in $A_i \cup B_i$. $\square$

Based on Lemmas 2 and 3, we define the notion of *large enough* weakly chorded ring as an infinite ring or as a ring which fulfills one of the two conditions of Lemma 3.

From this point on, we will always assume that the weakly chorded ring is large enough.

¿From Lemmas 1–3 it follows that in a large enough chorded ring, a dynamo of length k cannot contain in its window, $p + 1$ consecutive white nodes.

Theorem 1 *A dynamo of length k for $C\langle p, k\rangle$ cannot contain $p+1$ consecutive white nodes in its window.*

PROOF. By contradiction, assume that Y is a dynamo of length k that contains a block $x_i \ldots x_{i+p}$ of $p + 1$ white nodes in its window. Since outside the window all the nodes are white, Y contains a white block $R[i]$ if $C\langle p, k\rangle$ infinite, or $A_i \cup B_i$, if $C\langle p, k\rangle$ finite but large enough, as defined in Lemmas 2 and Lemma 3. By Lemma 1 this yields a contradiction. $\square$

We now characterize a pattern whose presence is *prescribed* inside the window of a dynamo of length k.

Theorem 2 *Let $C\langle p,k\rangle$ be a weakly chorded ring. In $C\langle p,k\rangle$, a dynamo of length k contains at least a bounded pattern P of $length(P) \leq 2p+1$ and $weight(P) \geq p+1$.*

PROOF. Let x be a node that becomes black due solely to the influence of the initial blacks (i.e., it becomes black at step 1). We now distinguish two cases depending on whether x is inside or outside the window.
Case 1: Let x be inside the window. Since the window has length k, its neighbors at distance k are outside the window and, hence, initially white. Thus, for x to become black, there must be $p+1$ blacks among its neighbors at distance $1, 2, \ldots, p$. It follows that the pattern of length $2p+1$ which has x at its center contains at least $p+1$ black nodes.
Case 2: Let x be outside the window. Let $w_1, \ldots, w_k$ be the nodes of the window. Assume, without loss of generality, that x is to the left of the window; thus, all its $p+1$ left neighbors are white. Since x must have at least $p+1$ black neighbors to become black, it follows that all its right neighbors must be black (and, thus, inside the window). In other words, x is adjacent to w_1, and $w_1, \ldots, w_p$ are all black. Consider now the next $p+1$ nodes $w_{p+1}, \ldots, w_{2p+1}$. By Theorem 1, a pattern of $p+1$ white nodes is forbidden inside the window of a dynamo; this implies that at least one of $w_{p+1}, \ldots, w_{2p+1}$ is black. Let w_j be such a node. Thus, the pattern P corresponding to $w_1, \ldots, w_j$ is such that $length(P) \leq 2p+1$ and $weight(P) \geq p+1$. $\square$

Let $W(X) = w_0, w_1, \ldots w_{k-1}$ be the window of a dynamo X of length k. From Theorem 2 we have that any dynamo X of length k must contain a bounded pattern with length at most $2p+1$ and weight at least $p+1$. Let $I(X) = w_{i(X)}, \ldots, w_{i(X)+q(X)-1}$ denote a such a pattern, where $q(X) \leq 2p+1$ denotes its length.

Lemma 4 *In $C\langle p,k\rangle$, for every dynamo X of length k:*

$$weight(X) \geq p+1+\lceil\frac{i(X)}{p+1}\rceil + \lceil\frac{k-i(X)-q(X)}{p+1}\rceil.$$

PROOF. By Theorem 1, a pattern of $p+1$ consecutive white nodes is forbidden in a dynamo of length k; it follows that the portion of the window $w_0, \ldots, w_{i(X)-1}$ preceding the pattern $I(X)$ must contain at least $\lceil\frac{i(X)}{p+1}\rceil$ blacks and the portion $w_{i(X)+q(X)}, \ldots, w_{k-1}$ following $I(X)$ must contain at least $\lceil\frac{k-i(X)-q(X)}{p+1}\rceil$ blacks. Since $I(X)$ contains at least $p+1$ blacks, it follows that $weight(X) \geq p+1+\lceil\frac{i(X)}{p+1}\rceil + \lceil\frac{k-i(X)-q(X)}{p+1}\rceil$. $\square$

We can now establish a lower bound on the weight of a dynamo of length k.

Theorem 3 *In $C\langle p,k\rangle$, for every dynamo X of length k:*

$$weight(X) \geq p+1+\lceil\frac{k-2p-1}{p+1}\rceil.$$

PROOF. From Lemma 4, it suffices to notice that, for all $i(X)$ $(0 \leq i(X) \leq k-q(X)-1)$, $p+1+\lceil\frac{i(X)}{p+1}\rceil+\lceil\frac{k-i(X)-q(X)}{p+1}\rceil \geq p+1+\lceil\frac{k-q(X)}{p+1}\rceil$. Moreover, for all $q(X)$ $(p+1 \leq q(X) \leq 2p+1)$, $p+1+\lceil\frac{k-q(X)}{p+1}\rceil \geq p+1+\lceil\frac{k-2p-1}{p+1}\rceil$. □

3.2 Optimal Dynamos

We now show that there exist dynamos of length k and, furthermore, their weight matches the lower bound established in the previous section. We first introduce a useful lemma.

Lemma 5 *The patterns $\mathbf{(1)}^p\mathbf{(0)}^i\mathbf{(1)}$ and $\mathbf{(1)(0)}^i\mathbf{(1)}^p$, $i \leq p$, are converging.*

PROOF. We now prove the Lemma by induction for block $\mathbf{(1)}^p\mathbf{(0)}^i\mathbf{(1)}$; the argument for block $\mathbf{(1)(0)}^i\ \mathbf{(1)}^p$ is symmetric.
Base case: $\mathbf{(1)}^p\mathbf{(0)(1)}$: the unique white node is connected to p black nodes to its left and at least one black node to its right; since it has at least $p+1$ black neighbors, it will also become black.
Inductive case: $\mathbf{(1)}^p\mathbf{(0)}^{i+1}\mathbf{(1)}$: let us assume that the proposition holds up to and including $i<p$; the leftmost white node of the pattern has exactly p blacks in its left neighborhood, and at least one black node in its right neighborhood (since $i+1 \leq p$); thus, it becomes black and the pattern becomes $\mathbf{(1)}^{p+1}\mathbf{(0)}^i\mathbf{(1)}$ which converges by inductive hypothesis. □

We now introduce the notion of initiating and filling pattern. We call *initiating pattern* for $C\langle p,k\rangle$ a converging bounded pattern of length at most $2p+1$ whose weight is at least $p+1$. An initiating pattern P is *maximum* if $length(P)=2p+1$ and $weight(P)=p+1$.

Lemma 6 *There exist maximum initiating patterns for any $C\langle p,k\rangle$.*

PROOF. Consider, for example, $\mathbf{(1)(0)}^p\mathbf{(1)}^p$. Such a pattern is converging by Lemma 5. Moreover, its weight is $p+1$ and its length is $2p+1$; hence, it is a maximum initiating pattern. □

We call *right (left) filling pattern* for $C\langle p,k\rangle$ a pattern which, when located to the right (left) of a pattern of length and weight p (i.e., $(\mathbf{1})^p$), is always converging. Examples of fillings patterns are given in the following lemma.

Lemma 7 *The patterns* $(\mathbf{0})^i(\mathbf{1})$ *and* $(\mathbf{1})(\mathbf{0})^i$*,* $i \leq p$*, are right and left filling, respectively.*

PROOF. Consider pattern $(\mathbf{0})^i(\mathbf{1})$ with $i \leq p$. When located to the right of a pattern of length and weight p we have the pattern $(\mathbf{1})^p\ (\mathbf{0})^i(\mathbf{1})$ which is converging by Lemma 5. The same argument holds for $(\mathbf{1})(\mathbf{0})^i$. $\square$

By definition of initiating and filling patterns, dynamos can be constructed using the following method:

General Construction

(1) Choose an arbitrary initiating pattern, place it anywhere in the length-k window.
(2) Complete the right (left) portion of the window with right (left) filling patterns.

Theorem 4 *The construction described above for* $C\langle p,k\rangle$ *forms a dynamo of length* k*.*

PROOF. By definition, the initiating pattern converges to a sequence of at least $p+1$ consecutive $\mathbf{1}$s; this enables the sequential convergence of the attached filling patterns. Hence the entire window becomes black; by Property 1, such configuration is converging and, thus, is a dynamo. $\square$

We now establish an upper bound on the weight of length-minimal dynamos; this bound matches the lower bound previously established.

Theorem 5 *There exist dynamos of length* k *for* $C\langle p,k\rangle$ *whose weight is* $p+1+\lceil\frac{k-2p-1}{p+1}\rceil$.

PROOF. It is sufficient to place any maximum initiating pattern at the beginning of the window and complete the window with $\lfloor\frac{k-(2p+1)}{p+1}\rfloor$ filling patterns of the form $(\mathbf{0})^p(\mathbf{1})$ and one filling pattern of the form
$(\mathbf{0})^{k-(2p+1)-(p+1)\lfloor\frac{k-(2p+1)}{p+1}\rfloor-1}(\mathbf{1})$. $\square$

From Theorems 3 and 5 we have that:

Theorem 6 *A dynamo X is* optimal *for weakly chorded $C\langle p,k\rangle$ iff: $length(X)=k$ and $weight(X)=p+1+\lceil \frac{k-2p-1}{p+1}\rceil$.*

3.3 Case study: Double Loop Networks

In this Section we give a complete characterization of the optimal dynamos for the class of *double loop networks*. These graphs are exactly the chorded rings $C\langle 1,k\rangle$, $k>1$, and coincide with the class of chordal rings of degree four.

The following is a corollary of Theorem 6:

Lemma 8 *A dynamo X is optimal in a double loop network iff $length(X)=k$ and $weight(X)=2+\lceil \frac{k-3}{2}\rceil=\lceil \frac{k+1}{2}\rceil$.*

The construction described in the proof of Theorem 5 allows to construct some optimal dynamos; in the case of double loop networks, we can actually characterize all the optimal dynamos (see Table 1).

Table 1
Windows of optimal dynamos for double loop networks.

Optimal Dynamos		
	Pattern	Number
k odd	$\mathbf{(10)}^{\lfloor \frac{k}{2}\rfloor}\ \mathbf{(1)}$	1
k even	$\mathbf{(10)}^{\alpha}\ \mathbf{(1)}\ \mathbf{(10)}^{\beta}\ \mathbf{(1)}$ with $\alpha+\beta=\frac{k-2}{2}$	$\lceil \frac{k}{2}\rceil$

Theorem 7 *In any $C\langle 1,k\rangle$: if k is odd the only optimal dynamo is $\mathbf{(10)}^{\lfloor \frac{k}{2}\rfloor}\mathbf{(1)}$; if k is even there are $\lceil \frac{k}{2}\rceil$ optimal dynamos and they have the form $\mathbf{(10)}^{\alpha}$ $\mathbf{(1)}$ $\mathbf{(10)}^{\beta}$ $\mathbf{(1)}$ with $\alpha+\beta=\frac{k-2}{2}$.*

PROOF. From Lemma 8, we know that a dynamo is optimal iff it contains exactly $\lceil \frac{k+1}{2}\rceil$ black nodes. Moreover, by Theorem 1, the presence of the pattern **(00)** is forbidden in its window. The all and only configurations verifying these two conditions correspond to the ones described in Table 1.

Finally, these configurations can be obtained by concatenating the maximum initiating pattern **(101)** with $\lfloor \frac{k-3}{2}\rfloor$ filling patterns of the form **(01)** and (if k is even) with one additional pattern of the form **(1)**; hence, by Theorem 4, they are dynamos. □

Notice that the double loop networks coincide with the class of tori with snake-like connections on the rows (torus cordalis). In [7,8], different types of tori (among which the torus cordalis) have been investigated and lower and upper bounds on the size of irreversible and monotone dynamos are given.

3.4 Case study: Triple Loop $C\langle 2, k\rangle$

We now study the triple loop network $C\langle 2, k\rangle$. Exact bounds on the weight of optimal dynamos for these graphs follow as a corollary of Theorem 6:

Lemma 9 *A dynamo X is* optimal *in a triple loop $C\langle 2, k\rangle$ iff:*
$length(X) = k$ and $weight(X) = 3 + \lceil \frac{k-5}{3} \rceil = \lceil \frac{k+4}{3} \rceil$.

The construction described in the proof of Theorem 5 allows to construct some optimal dynamos; in the following we will actually provide a complete characterization. To do so, we introduce a different problem which will be shown to be equivalent to the problem of constructing all optimal dynamos for $C\langle 2, k\rangle$ chordal rings.

More precisely, instead of finding directly the set of optimal dynamos, say OD, we introduce a problem that is easier to solve (the Ring Problem) and we design a transformation Ψ from its solutions S (described in Lemma 10), to a set of configurations $\Psi(S)$ (described in Theorem 8). We then show (Theorem 13) that $\Psi(S)$ coincides with OD.

The sketch of our reasoning leading to Theorem 13 is the following: We first prove that every configuration of $\Psi(S)$ is an optimal dynamo (Theorem 10) by showing that: i) all the elements of $\Psi(S)$ satisfy some structural properties (Lemma 11); ii) every element satisfying these properties is a dynamo (Theorem 9); iii) every string of $\Psi(S)$ satisfies the conditions of optimality of Lemma 9. We then prove that every optimal dynamo is in $\Psi(S)$ (Theorem 12), showing that the reverse transformation Ψ^{-1} applied to any optimal dynamo is a solution to the Ring Problem, based on the fact that: i) every dynamo verifies the structural properties used before (Theorem 11), and ii) every optimal dynamo satisfies the length and weight conditions given in Lemma 9.

The Ring Problem: *Given a ring where every node is white, color black the minimum number of nodes in such a way that no three consecutive nodes are white.*

Clearly, any solution to this problem on a ring of size m requires $\lceil \frac{m}{3} \rceil$ blacks. It is not difficult to determine all the configurations solving the ring problem, indicated in the following.

Lemma 10 *Let R_m denote the set of solutions to the ring problem. Then, R_m is composed of the following circular strings:*

(1) if $m = 3j$: $\mathbf{(001)}^j$;
(2) if $m = 3j+1$: $\mathbf{(1)(001)}^j$, and $\mathbf{(01)(001)}^a\mathbf{(01)(001)}^b$, with $a+b = j-1$;
(3) if $m = 3j+2$:$\mathbf{(001)}^a\mathbf{(01)(001)}^b$, with $a+b = j$.

Let us now define a transformation Ψ from the elements of R_m to linear strings of length $m+2$. Consider a circular string $X =< x_1, \ldots, x_m >\in R_m$; let x_i be an arbitrary black element, and let $< x_i, x_{i+1}, \ldots, x_m, x_1, x_2, \ldots, x_i >=< y_1, \ldots, y_{m+1} >= Y$ be the linear string of length $m+1$ obtained from X by starting from x_i and replicating it at the end of the obtained string. Let y_j be a black element of Y, and let $< y_1, \ldots, y_{j-1}, y_j, y_j, y_{j+1}, \ldots, y_{m+1} >$ be the linear string of size $m+2$ obtained from Y by duplicating y_j.

Let $\Psi(R_m)$ denote the set of all the linear strings obtained from the elements of R_m through the above construction. We can actually completely characterize the elements of the set.

Table 2
The set $\Psi(R_m)$ where $k = m+2$.

k	String	Form
$3i+2$	$\mathbf{(100)}^\alpha\mathbf{(11)(001)}^\beta$	(1)
	$\alpha+\beta = i$	
$3i+1$	$\mathbf{(100)}^\alpha\ \mathbf{(10)}\ \mathbf{(100)}^\beta\ \mathbf{(11)}\ \mathbf{(001)}^\gamma$	(2)
	$\mathbf{(100)}^\alpha\ \mathbf{(11)}\ \mathbf{(001)}^\beta\ \mathbf{(01)}\ \mathbf{(001)}^\gamma$	(3)
	$\alpha+\beta+\gamma = i-1$	
$3i$	$\mathbf{(100)}^\alpha\ \mathbf{(11)}\ \mathbf{(001)}^\beta\ \mathbf{(1)(001)}^\gamma$	(4)
	$\mathbf{(100)}^a\mathbf{(10)}\ \mathbf{(100)}^b\ \mathbf{(10)(100)}^c\ \mathbf{(11)}\ \mathbf{(001)}^d$	(5)
	$\mathbf{(100)}^a\mathbf{(10)}\ \mathbf{(100)}^b\ \mathbf{(11)(001)}^c\ \mathbf{(01)}\ \mathbf{(001)}^d$	(6)
	$\mathbf{(100)}^a\mathbf{(11)}\ \mathbf{(001)}^b\ \mathbf{(01)(001)}^c\ \mathbf{(01)}\ \mathbf{(001)}^d$	(7)
	$a+b+c+d = i-2$	
	$\alpha+\beta+\gamma = i-1$	

Theorem 8 *The set $\Psi(R_m)$ is composed of the strings shown in Table 2 where k=m+2.*

PROOF. *Case $m = 3j$*: The initial linearization and duplication of a black leads to the linear string $\mathbf{(1)}\ \mathbf{(001)}^j$. The duplication of an arbitrarily chosen black produces a string of form (1).

Case $m = 3j + 1$: The initial linearization and duplication of a black leads either to the linear string $\mathbf{(100)}^a\ \mathbf{(11)}\ \mathbf{(001)}^b$ with $a + b = j$, or to the linear string $\mathbf{(1)}\ \mathbf{(001)}^a\ \mathbf{(01)}\ \mathbf{(001)}^b\ \mathbf{(01)}\ \mathbf{(001)}^c$ with $a + b + c = j - 1$.
Subcase $\mathbf{(100)}^a\ \mathbf{(11)}\ \mathbf{(001)}^b$ *with* $a + b = j$: The duplication of an arbitrarily chosen black produces one of the following three types of strings:

- $\mathbf{(100)}^c\ \mathbf{(1100)}\ \mathbf{(100)}^d\ \mathbf{(11)}\ \mathbf{(001)}^b$ with $c + d = a - 1$, form (4);
- $\mathbf{(100)}^a\ \mathbf{(111)}\ \mathbf{(001)}^b$, form (4);
- $\mathbf{(100)}^a\ \mathbf{(11)}\ \mathbf{(001)}^c\ \mathbf{(0011)}\ \mathbf{(001)}^d$ with $c + d = b - 1$, form (4).

Subcase $\mathbf{(1)}\ \mathbf{(001)}^a\ \mathbf{(01)}\ \mathbf{(001)}^b\ \mathbf{(01)}\ \mathbf{(001)}^c$ *with* $a + b + c = j - 1$: The duplication of an arbitrarily chosen black produces one of the following six types of strings:

- $\mathbf{(11)}\ \mathbf{(001)}^a\ \mathbf{(01)}\ \mathbf{(001)}^b\ \mathbf{(01)}\ \mathbf{(001)}^c$, form (7);
- $\mathbf{(1)}\ \mathbf{(001)}^d\ \mathbf{(0011)}\ \mathbf{(001)}^e\ \mathbf{(01)}\ \mathbf{(001)}^b\ \mathbf{(01)}\ \mathbf{(001)}^c$ with $d + e = a - 1$, form (7);
- $\mathbf{(1)}\ \mathbf{(001)}^a\ \mathbf{(011)}\ \mathbf{(001)}^b\ \mathbf{(01)}\ \mathbf{(001)}^c$, form (6);
- $\mathbf{(1)}\ \mathbf{(001)}^a\ \mathbf{(01)}\ \mathbf{(001)}^d\ \mathbf{(0011)}\ \mathbf{(001)}^e\ \mathbf{(01)}\ \mathbf{(001)}^c$ with $d + e = b - 1$, form (6);
- $\mathbf{(1)}\ \mathbf{(001)}^a\ \mathbf{(01)}\ \mathbf{(001)}^b\ \mathbf{(011)}\ \mathbf{(001)}^c$, form (5);
- $\mathbf{(1)}\ \mathbf{(001)}^a\ \mathbf{(01)}\ \mathbf{(001)}^b\ \mathbf{(01)}\ \mathbf{(001)}^d\ \mathbf{(0011)}\ \mathbf{(001)}^e$ with $d + e = c - 1$, form (5).

Case $m = 3j + 2$: Duplicating a first black gives $\mathbf{(1)}\ \mathbf{(001)}^a\ \mathbf{(01)}\ \mathbf{(001)}^b$ with $a + b = j$. The duplication of an arbitrarily chosen black produces one of the following four types of strings:

- $\mathbf{(11)}\ \mathbf{(001)}^a\ \mathbf{(01)}\ \mathbf{(001)}^b$, form (3);
- $\mathbf{(1)}\ \mathbf{(001)}^c\ \mathbf{(0011)}\ \mathbf{(001)}^d\ \mathbf{(01)}\ \mathbf{(001)}^b$ with $c + d = a - 1$, form (3);
- $\mathbf{(1)}\ \mathbf{(001)}^a\ \mathbf{(011)}\ \mathbf{(001)}^b$, form (2);
- $\mathbf{(1)}\ \mathbf{(001)}^a\ \mathbf{(01)}\ \mathbf{(001)}^c\ \mathbf{(0011)}\ \mathbf{(001)}^d$ with $c + d = b - 1$, form (2). □

All these strings share some basic properties:

Lemma 11 *Given* $Z \in \Psi(R_m)$,

- *Z has length* $m + 2$
- $z_1 = z_{m+2} = 1$
- *Z does not contain any block* $\mathbf{(000)}$
- *Z contains a block* $\mathbf{(11)}$.

PROOF. By construction. □

Our interest in the ring problem, its solutions, the transformation, and the resulting strings rests in the fact that every element of $\Psi(R_m)$ is an optimal dynamo for $C\langle 2, m+2\rangle$. To prove this we first need to establish the following characterization.

Theorem 9 *Any configuration with a window of length k which contains two consecutive black nodes and which does not contain three consecutive white nodes is a dynamo for $C\langle 2, k\rangle$.*

PROOF. Let $W(X) = x_1 \dots x_k$ be the window of such a configuration and let, without loss of generality, $x_i = 1$ and $x_{i+1} = 1$; we shall consider two cases depending on i.
Case 1, $i \leq k-4$: Since $x_{i+2}x_{i+3}x_{i+4}$ cannot be a block of three consecutive whites, the pattern $x_i x_{i+1} x_{i+2} x_{i+3} x_{i+4}$ has length 5 and weight ≥ 3. Moreover, it is easy to verify that no matter which of the nodes $x_{i+2}x_{i+3}x_{i+4}$ is black, the pattern is converging; thus, it is an initiating pattern. So, X contains an initiating pattern. Since a block of three consecutive whites is forbidden, the portion of the window at the left of the pattern $x_i x_{i+1} x_{i+2} x_{i+3} x_{i+4}$ can contain only blocks of the form **(1)**, **(10)**, and **(100)** which are all left filling patterns (Lemma 7) and the portion of the window at the right of X can contain only blocks of the form **(1)**, **(01)**, and **(001)** which are all right filling patterns. Thus X is a dynamo.
Case 2, $i > k-4$: The proof is similar to the previous case by first showing that $x_{i-4}x_{i-3}x_{i-2}x_{i-1}x_i$ (instead $x_i x_{i+1} x_{i+2} x_{i+3} x_{i+4}$) is an initiating pattern and then proceeding as above. $\square$

We can now prove that transforms of the ring problem solutions are actually *optimal* dynamos.

Theorem 10 *Every element of $\Psi(R_{k-2})$ is a window of an optimal dynamo for $C\langle 2, k\rangle$.*

PROOF. By Lemma 11 and Theorem 9, it follows that the strings in $\Psi(R_{k-2})$ are windows of dynamos of length k. Moreover, it is easy to verify that their weight is $3 + \lceil \frac{k-5}{3} \rceil$; thus, by Lemma 9, they are optimal dynamos. $\square$

Our interest in the ring problem is further justified because *only* the strings $\Psi(R_{k-2})$ are windows of optimal dynamos for $C\langle 2, k\rangle$. To prove this, we will need to establish first a necessary condition for dynamos of length k (and, thus, for optimal dynamos).

Theorem 11 *Any dynamo of length k in $C\langle 2,k\rangle$ contains a pattern with two consecutive black nodes and does not contain any pattern of three consecutive whites.*

PROOF. Let X be a dynamo of length k. The fact that three consecutive white nodes are forbidden in its window $x_0 \dots x_{k-1}$ follows from Theorem 1. We now show by contradiction that X must contain two consecutive blacks in its window. Let us assume that X does not contain two consecutive blacks and we will show that the white nodes of X would form a white block, that is, a block where every white node is connected to four other white nodes.
Let us consider an arbitrary white node x_i inside the window $(1 \leq i \leq k-2)$. Node x_i is connected to the white nodes x_{i+k} and x_{i-k} outside the window (the ring is large enough); x_i is also connected to nodes x_{i-1} and x_{i-2} on its left (one of which must be white since we have assumed that there are no two consecutive blacks) and x_{i+1}, x_{i+2} on its right (one of which must be white). Thus, an arbitrary white node x_i inside the window has four white neighbors.
Let us now consider an arbitrary node x_j outside the window. In this situation we will distinguish three cases:
Case 1, $j \in \{-1, k\}$: Node x_{-1} is connected to four nodes: x_1 which cannot be black (because x_0 is black), x_{-2}, x_{-3}, and x_{-1-k} (because the ring is large enough). The same holds symmetrically for x_k.
Case 2, $j \in \{-2, k+1\}$: Node x_{-2} is connected to four white nodes: x_{-1}, x_{-3}, x_{-4}, and x_{-2-k} (because the ring is large enough). The same holds symmetrically for x_{k+1}.
Case 3, $j < -2$ or $j > k+1$: Node x_j is connected to four white nodes: $x_{j\pm 1}$ and $x_{j\pm 2}$. Thus, the set of white nodes of X forms a white block. The existence of a white block in a dynamo contradicts Lemma 1, which completes the proof. □

Thus, to prove that the strings in $\Psi(R_{k-2})$ are the only optimal dynamos for $C\langle 2,k\rangle$, it suffices to show that only those strings satisfy the properties of Lemma 11. In other words, every string of length k, starting and ending with a black, containing two consecutive black nodes, and not containing three consecutive white nodes is the transform of a solution to the ring problem of size $k-2$.

Theorem 12 *The window of every optimal dynamo for $C\langle 2,k\rangle$ is an element of $\Psi(R_{k-2})$.*

PROOF. Let $W(X) = x_1, \dots, x_k$ be the window of an optimal dynamo for $C\langle 2,k\rangle$. By Theorem 11 X contains at least two consecutive blacks, say x_i and x_{i+1}, and does not contain three consecutive whites.

Let $Y = x_1, \ldots, x_i, x_{i+2}, \ldots, x_k$ be the string obtained from x by removing x_{i+1}; then also Y does not contain three consecutive whites.
Moreover, since both x_1 and x_k are black, it follows that also the *circular* string $Z =< x_1, \ldots, x_i, x_{i+2}, \ldots, x_{k-1} >$ does not contain three consecutive whites.
Since X is an optimal dynamo, by Lemma 9 $weight(X) = \lceil \frac{k+4}{3} \rceil$; it follows that the number of blacks in Z is $\lceil \frac{k+4}{3} \rceil - 2 = \lceil \frac{k-2}{3} \rceil$ which means that Z is a solution to the ring problem. In other words, $W(X) \in \Psi(R_{k-2})$. $\square$

Summarizing, by Theorems 10 and 12, we have:

Theorem 13 *The windows of optimal dynamos for $C\langle 2, k\rangle$ are all and only the strings shown in Table 2.*

4 Strongly Chorded Rings

A *strongly chorded ring* is a chorded ring $C\langle p, k\rangle$ such that $p \geq \lfloor \frac{k}{2} \rfloor$. In the following let $q = k-p$, so the chordal ring can be also indicated as: $C\langle k-q, k\rangle$. In this section we show that the trivial lower bound $p+1$ (see Corollary 1) on the weight of dynamos is matched by the optimal dynamos for this class of chorded rings. In the following we shall distinguish the two cases: $k > p+1$ and $k \leq p+1$.

4.1 Optimal Dynamos - Case $k > p+1$

We now consider strongly chorded rings $C\langle p, k\rangle$ where $k > p+1$. (Recall that $q = k - p$.)

For the next results, it is useful to decompose the length-k window of a dynamo as follows: Z_c is the part of the window from which any other node of the window can be seen, including the two extreme blacks, it is thus composed of the central $k - 2(q-1)$ nodes; Z_l and Z_r are the (possibly empty) sets of nodes located respectively at the left and at the right of Z_c, excluding the two extreme blacks (see Figure 1). Observe that Z_c is never empty since $q \leq \lceil \frac{k}{2} \rceil$; on the other hand, Z_l and Z_r are empty only when $q = 2$.

Lemma 12 *In a length-k dynamo for $C\langle p, k\rangle$, all white nodes of Z_c become black in one step.*

PROOF. Since, for all i such that $0 \leq i \leq k - 2(q-1) - 1$, we have $1 + (q -$

$$1\underbrace{\overbrace{\bullet\cdots\bullet}^{Z_l}}_{q-2}\underbrace{\overbrace{\bullet\cdots\bullet}^{Z_c}}_{k-2(q-1)}\underbrace{\overbrace{\bullet\cdots\bullet}^{Z_r}}_{q-2}1.$$

Fig. 1. Decomposition of the length-k window of a strongly chorded ring dynamo

$2)+i \leq k-q$, every white node v of Z_c is connected to all the nodes in the window. Thus, v is connected to at least $p+1$ black nodes inside the window of the dynamo. Hence, it becomes black. □

Using the previous lemma we now show a dynamo matching the lower bound.

Theorem 14 *In strongly chorded rings $C\langle p,k\rangle$ with $k > p+1$, there exist dynamos of length k with weight $p+1$.*

PROOF. We consider two cases depending on whether $|Z_l|+|Z_r| \leq p-1$ or not.
Case 1, $2(k-p-2) \leq p-1$: To obtain a dynamo, we color black all nodes of $Z_l \cup Z_r$ and $p-1-2(k-p-2)$ nodes of Z_c arbitrarily chosen. In total there are $p+1$ black nodes. By Lemma 12 the nodes in Z_c become black in one step; so, the window converges and by Property 1 our construction is a dynamo.
Case 2, $2(k-p-2) \geq p-1$: To obtain a dynamo, we color black the $\lfloor \frac{p-1}{2} \rfloor$ nodes to the left of Z_c and the $\lceil \frac{p-1}{2} \rceil$ nodes to the right of Z_c. Also in this case we have a total of $p+1$ black nodes and, by Lemma 12 the nodes in Z_c become black in one step. Thus, there will be at least $3p+3-k$ consecutive black nodes after the first step: the $k-2(k-p-1)$ nodes of Z_c plus the $\lfloor \frac{p-1}{2} \rfloor$ to its left and the $\lceil \frac{p-1}{2} \rceil$ to its right. Consider the closest white node v to the left of this set S of consecutive black nodes with *step* ≤ 1. Node v is at distance at least $q-2-\lfloor \frac{p-1}{2} \rfloor$ from the leftmost black node of the window. But it is easy to verify that $q-2-\lfloor \frac{p-1}{2} \rfloor \leq p$; hence, v has p black neighbors after the first step on its right and at least one black on its left, thus it becomes black in at most two steps. By symmetry, also the closest white node v' to the right of S becomes black in at most 2 steps. Applying this reasoning inductively, it follows that the entire window converges. □

Thus, we have that:

Theorem 15 *A dynamo X is* optimal *for strongly chorded $C\langle p,k\rangle$ iff: $length(X)=k$ and $weight(X)=p+1$.*

We now examine properties of optimal dynamos for strongly chorded rings. The next lemma gives a necessary condition on the weight of an optimal dynamo.

Lemma 13 *In an optimal dynamo, $weight(Z_c) < k - 2(q-1)$.*

PROOF. By contradiction, assume that $weight(Z_c) = k - 2(q-1)$, that is, all the initially white nodes in the window belong to $Z_l \cup Z_r$. Let v be an arbitrary white node of Z_l. Since $k - q < k - 2(q-1) + (q-2) + 1$, node v is not connected to the rightmost black node of the window; hence, it cannot become black in the first step being connected to less than $p+1$ black nodes. Hence, the white node in Z_l will not become black; obviously neither will any white node to the left of the window. Symmetrically, no node of Z_r will become black in the first step and neither will the white nodes to the right of the window. $\square$

Although necessary and sufficient for $k = p+1, k = p+2, \cdots, k = p+5$, this condition is not sufficient in general. For example, **(11001111101110011)** is not a dynamo of $C\langle 11, 17\rangle$.

Obviously, a node of Z_l cannot "see" (i.e., be connected to) all of Z_r, and vice-versa. Let x be a node of $Z_l \cup Z_r$. We denote by $invisible(x)$, the part of $Z_l \cup Z_r$ that x cannot see. We define $B(x) = weight(invisible(x))$ and $B = \min\{B(x) \mid x \in Z_l \cup Z_r \wedge x \text{ is white}\}$. Finally, we define the distance of a node $x \in Z_l \cup Z_r$ to Z_c, $d(x, Z_c)$, as its distance to the closest node of Z_c on the external ring; $\forall i, x_i$ and x_{i+1} are at distance 1 from each other. Notice that, $B(x)$ increases with $d(x, Z_c)$.

We can now state the next result which gives a condition under which no white node ever turns black.

Lemma 14 *If $B > weight(Z_l \cup Z_r) - q + 2$, no white of $Z_l \cup Z_r$ changes its color.*

PROOF. Let x be a white node of Z_l. This node sees the leftmost black node of the window, as well as all black nodes of Z_c and at most $weight(Z_l \cup Z_r) - B(x)$ black nodes in $Z_l \cup Z_r$. By Lemma 12, we can consider that all the nodes of Z_c have turned black. Using the hypothesis and the fact that $\forall x, B(x) \geq B$, we can conclude that node x sees at most $1 + k - 2(q-1) + weight(Z_l \cup Z_r) - B(x) < p+1$ black nodes in its neighborhood; hence, it cannot become black in the first step. Hence, the white node in Z_l will not become black; obviously neither will any white node to the left of the window. Symmetrically, no node of Z_r will become black in the first step and neither will the white node to the right of the window. $\square$

A trivial corollary of this result is the following:

Corollary 3 *The closest white node to Z_c, f, changes its color iff $B(f) \leq weight(Z_l \cup Z_r) - q + 2$.*

Unfortunately, the result of this corollary does not imply that other white nodes could become black. However, the next theorem gives a stronger condition entailing all white nodes to eventually recolor themselves.

Theorem 16 *If every white node x of $Z_l \cup Z_r$ is such that $2d(x, Z_c) \geq q$, then all white nodes eventually become black.*

PROOF. In order to prove this, we need to rank all white nodes of $Z_l \cup Z_r$ according to their distance to Z_c, and consider them each in turn iteratively. We also consider that all white nodes of Z_c have already turned black, by Lemma 12. Without loss of generality, assume $x \in Z_l$. Node x sees the leftmost black node of the window, since $1 + q - 2 - d(x, Z_c) \leq k - q$. To the right, the number of black nodes is: $(d(x, Z_c) - 1) + k - 2(q - 1) + (d(x, Z_c) - 1)$ (respectively: left of Z_c, Z_c, right of Z_c), which is $\geq k - q$. Hence, x see sufficiently many black nodes in its neighborhood to turn black. When x turns black, the number of black nodes around Z_c increases, which allows the process to repeat itself until all nodes have turned black. $\square$

Notice that, case 2 of Theorem 14 is actually a consequence of this result. The complete characterization of dynamos for such chorded rings remains an open problem.

4.2 Optimal Dynamos - Case $k = p + 1$

In this section we consider strongly chorded ring $C\langle p, k\rangle$ where $k = p + 1$. These chordal rings $C\langle k - 1, k\rangle$ are also called *fan graphs*. Notice that, when $k = \lfloor \frac{n}{2} \rfloor$ such a chordal ring is a complete graph.

We now show that the trivial lower bound on the weight of an optimal dynamo is matched by exactly one dynamo.

Since the degree of this type of chordal rings is $2k$, the trivial lower bound on the weight of an optimal dynamo is k.

Clearly there exists a dynamo of length and weight k: the window consisting of a sequence of k black nodes (such a configuration is a dynamo by Property 1); and obviously there is no other dynamo with length and weight k. Thus we have:

Theorem 17 *A configuration X is an optimal dynamo for strongly chorded $C\langle k-1,k\rangle$ iff: $length(X) = weight(X) = k$. There exists exactly one such an optimal dynamo.*

5 Concluding remarks

In this paper we have studied the effect of a simple majority rule in two classes of chordal rings: *weakly chorded* and *strongly chorded* rings.

More precisely, we have concentrated on irreversible dynamos in such topologies (i.e. initial configurations which lead the system to a monochromatic fixed point) and we have studied optimal dynamos (i.e., dynamos of minimum length containing the minimal number of blacks).

After establishing some basic bounds on the weight and length of dynamos for general chordal rings, we have shown a lower bound on the weight of dynamos of minimal length for large enough weakly chorded rings and we have proved that such a bound is tight giving a constructive upper bound. For the particular case of the well known class of double and triple loop networks, we have provided a complete characterization of the optimal dynamos. Also for large enough strongly chorded rings, we have established tight, constructive bounds.

There are several problems still open and research direction still unexplored.

First of all, our results on weakly chorded rings hold when they are large enough. It would be interesting to find tight bounds for small chorded rings as well, when wrap-around effects becomes relevant and the dynamics more complex. It is not difficult to see that Lemma 2, and, thus, Theorem 1 can be adapted to finite and possibly small weakly chorded rings provided that the longest chord divides the ring size. As a consequence, our results would hold also in such a case; for all other cases, the problem is still open.

Another interesting direction would be the study of other majority rules and mending conditions on the initial faults (blacks); more precisely, in this paper we have considered irreversible dynamics, i.e., situation where an initially black node cannot change its color to white. Under this assumption, black nodes describe permanent faults in a system and the dynamics describes their propagation. On the other hand, it would be interesting to analyze the case when also the initially black nodes could change their color; this case corresponding to transient faults.

Finally, it would be interesting to look at the propagation time of dynamos in the case of synchronous dynamics. For all the dynamos considered in this paper the propagation time of a synchronous execution would be $O(n)$, where n is the number of nodes of the chordal ring. However, instead of concentrating on minimum size dynamos regardless of the propagation time, it would be interesting to study the trade-offs between number of black nodes in a dynamo and its propagation time.

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