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BACKWARD CONSISTENCY AND SENSE OF DIRECTION IN ADVANCED DISTRIBUTED SYSTEMS*

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Abstract. Studies on the relationship between label consistency, computability, and complexity assume the existence of *local orientation*; this assumption is in fact at the basis of the *point-to-point* model and is realistic for systems where a communication link can connect only two entities. However, in systems which use more advanced communication and interconnection technology, such as *buses*, *optical* networks, and *wireless* communication media, and more importantly, in heterogeneous systems (such as the *Internet*) which include any combination of the above, local orientation *cannot* be assumed. This implies that the entire established body of results on the relationship between label consistency (e.g., *sense of direction*) and computability and complexity does not hold for systems with advanced communication technology.

In this paper we consider a new type of consistency which we shall call *backward consistency* and which, unlike sense of direction, can exist even without local orientation. Thus, unlike all previous forms of consistency, it can be found (or designed) in advanced distributed systems.

We study backward consistency both in terms of its relationship with the traditional properties of local orientation and (weak) sense of direction, and with respect to symmetries of the edge labelings and of the naming functions. We show that backward consistency is computationally equivalent to sense of direction; in other words, it is possible to take advantage of the computational power of sense of direction even in the absence of local orientation.

Key words. distributed computing, sense of direction, global consistency

AMS subject classifications. 05C78, 68Q10, 68R10, 68W15

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1. Introduction.

A distributed system is a collection of computational entities communicating by exchanging messages. Depending on how the communication is achieved, different models exist. In the *point-to-point* model, the communication topology of the system is viewed as an edge-labeled undirected graph $(G = (V, E), \lambda)$ where nodes correspond to the system entities, edges represent pairs of neighboring entities (i.e., entities which can communicate directly), each node $x \in V$ has a local label (usually called port number) $\lambda_x(\langle x, y\rangle)$ associated to each of its incident edges $\langle x, y\rangle$, and $\lambda = \{\lambda_x : x \in V\}$ is the set of labeling functions.

In studies on computing in distributed systems, recurring challenging questions arise concerning the relationships between *local views* and *global consistency*, and concerning their impact on distributed computability and communication complexity. In these studies, *locality* is at the entity level: the local view of an entity x consists of just the labeling λ_x of its communication ports (or communication links, incident edges) and, sometimes, of the naming β_x used to refer to the other entities. A major goal of this investigation is to derive under what properties of λ and/or β it is possible to

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infer global properties about the entire system. A practical aim is to determine if and how these properties can be usefully employed, e.g., to yield more efficient distributed computations. Examples of these studies include *compact routing,* which focuses on determining under what conditions the local port numbering provides global routing information for given global naming schemes (see [25] for a recent survey), and *sense of direction,* which analyzes the impact on communication complexity of local port labelings and local naming schemes (see [21] for a recent survey).

In particular, there is a large body of evidence on the positive impact on complexity of the set of global consistency constraints satisfied by labelings with sense of direction (e.g., see [18, 19, 22, 30, 32, 33, 41]). Properties of sense of direction have been studied, for example, in [5, 8, 10, 11, 17, 20, 23, 24, 26, 42]. Several levels of sense of direction have been considered and investigated, with researchers trying to identify the "smallest" amount of consistency really needed (e.g., to efficiently solve a specific problem).

All studies in the relevant literature, with very few exceptions, make the basic (but often unstated) fundamental assumption that the entities are able to distinguish among their incident links, i.e., the labelings λ_x are injective functions. Even if the links haven't yet been assigned a label, it is assumed that an entity is capable of such an assignment (e.g., [15, 27, 31]). This assumption, usually called *local orientation*, is in fact at the basis of the *point-to-point* model and is realistic for systems where a communication link can connect only two entities.

If we are to model systems which use more advanced communication and interconnection technology, such as *buses*, *optical* networks, and *wireless* communication media, and more importantly, in heterogeneous systems (such as the *Internet*) which include any combination of the above, local orientation *cannot* be assumed. This is because any direct connection between k entities will correspond, at each of those entities, to $k-1$ edges with the same label; hence, if $k > 2$ (e.g., a bus), λ is not injective. That is, unless the system is point-to-point (i.e., each connection is between only two entities) there are nodes that cannot distinguish between some of their incident edges. This implies that the entire established body of results on the relationship between consistency, computability, and complexity does not hold and, thus, cannot be applied outside the point-to-point model. In other words, very little is known on this subject for systems with advanced communication technology. (A vast body of knowledge does indeed exist on these systems, but not on these topics.)

There are a few exceptions: the early work on computing in complete graphs with "ambiguous" labelings [26, 40], the more recent studies on computability in anonymous "wireless" networks [14, 37], and the results on leader election and function evaluation in anonymous networks with different types of "port awareness" [9, 45]. Note that "ambiguous," "wireless," and "port aware" all denote the (possible) absence of local orientation.

In this paper we consider a new type of consistency which we shall call *backward consistency.* Backward consistency is strongly related to the classical "forward" consistency implied by sense of direction; however, unlike sense of direction, it can exist even without local orientation. Thus, unlike all previous forms of consistency, backward consistency can be found (or designed) in advanced distributed systems.

In the following, we study backward consistency both in terms of its relationship with the traditional properties of local orientation and (weak) sense of direction, and with respect to symmetries of the edge labelings and of the naming functions. In particular, we focus on the conditions under which a labeling can have simultaneously forward and backward consistency.

We then consider the power of backward consistency and prove that *backward consistency is computationally equivalent to sense of direction.* In other words, it is possible to take advantage of the computational power of sense of direction even in the absence of local orientation. This counterintuitive result somehow extends and generalizes the results of [1, 14] on the insensitivity of ring networks to local orientation. In addition to the theoretical result, we also provide a simple effective procedure to efficiently map protocols exploiting sense of direction into ones using "backward sense of direction."

In the following we consider the undirected case (i.e., bidirectional communication capabilities); this is done only for simplicity of exposition, as all results extend to and hold in the directed case as well.

2. Basic definitions and properties.

2.1. Labelings and sense of direction. In this section we first recall the basic definitions of consistency, symmetry, and sense of direction in labeled graphs; most of the terminology follows [20].

Let $G = (V, E)$ be a simple connected undirected graph, and let $E(x)$ denote the set of edges incident to node $x \in V$. Given $G = (V, E)$ and a set Σ of labels, a *local labeling function* of $x \in V$ is any function $\lambda_x : E(x) \to \Sigma$ which associates a label $l \in \Sigma$ to each edge $e \in E(x)$. A set $\lambda = \{\lambda_x : x \in V\}$ of local labeling functions will be called a *labeling* of G, and by (G, λ) we shall denote the corresponding *(edge-)labeled graph.* If all functions in λ are injective, we say that λ is a *local orientation.* Note that local orientation is assumed in the point-to-point model and in almost all literature on informative labelings, including those on unlabeled networks (e.g., [15, 27, 31]). A labeling λ is *symmetric* if there exists a bijection $\psi : \Sigma \to \Sigma$ such that for each $\langle x, y\rangle \in E$, $\lambda_y(\langle y, x\rangle) = \psi(\lambda_x(\langle x, y\rangle))$; ψ will be called the edge-symmetry function.

A *walk* π in G is a sequence of edges in which the endpoint of one edge is the starting point of the next edge. Let $P[x]$ denote the set of all walks starting from $x \in V$, and let $P[x, y]$ denote the set of walks starting from $x \in V$ and ending in $y \in V$. Let $\Lambda_x : P[x] \to \Sigma^+$ and $\Lambda = \{\Lambda_x : x \in V\}$ denote the extension of λ_x and λ, respectively, from edges to walks; let $\Lambda[x] = \{\Lambda_x(\pi) : \pi \in P[x]\}$, and let $\Lambda[x, y] = \{\Lambda_x(\pi) : \pi \in P[x, y]\}$.

Given an edge-symmetry function ψ, we shall denote by $\Psi : \Sigma^+ \to \Sigma^+$ its extension to strings; i.e., for $\alpha = a_1 \cdot a_2 \cdot \ldots \cdot a_p \in \Sigma^+$, $\Psi(\alpha) = \psi(a_p) \cdot \ldots \cdot \psi(a_1)$, where $\cdot$ denotes the concatenation operator.

Informally, a labeled graph (G, λ) has *sense of direction* when it is possible to understand, from the labels associated to the edges, whether different walks from any given node x end in the same node or in different ones.

A *coding* function of (G, λ) is any function $\mathbf{c}$ with domain Σ^+. It is said to be *consistent* if the following holds $\forall x, y, z \in V$, $\forall \pi_1 \in P[x, y]$, $\pi_2 \in P[x, z]$: $\mathbf{c}(\Lambda_x(\pi_1)) = \mathbf{c}(\Lambda_x(\pi_2))$ iff $y = z$. In other words, the labelings of the walks *originating* from the same node are mapped to the same value iff they end in the same node. Thus, we can define a consistent coding if the following holds $\forall x, y, z \in V$, $\forall \alpha \in \Lambda[x, y]$, $\beta \in \Lambda[x, z]$: $\mathbf{c}(\alpha) = \mathbf{c}(\beta)$ iff $y = z$. We shall denote by $\mathcal{N}(\mathbf{c})$ the codomain of $\mathbf{c}$.

DEFINITION 1 (weak sense of direction). *A labeled graph* (G, λ) *has* weak sense of direction $\mathbf{c}$ *iff* $\mathbf{c}$ *is a consistent coding function of* (G, λ). *Alternatively, we shall say that* $\mathbf{c}$ *is a weak sense of direction in* (G, λ).

We see immediately that consistency requires local orientation, as follows.

LEMMA 1. *Let* (G, λ) *have weak sense of direction* $\mathbf{c}$. *Then* λ *has local orientation.*

Proof. By contradiction, let there exist two distinct neighbors y, z of a node $x \in V$ such that $\lambda_x(\langle x,y\rangle) = \lambda_x(\langle x,z\rangle)$; thus, $\mathbf{c}(\lambda_x(\langle x,y\rangle)) = \mathbf{c}(\lambda_x(\langle x,z\rangle))$. On the other hand, by definition of the consistent coding function, $\mathbf{c}(\lambda_x(\langle x,y\rangle)) \neq \mathbf{c}(\lambda_x(\langle x,z\rangle))$, which leads to the contradiction. ∎

Given a coding function $\mathbf{c}$, a *decoding function* $\mathbf{d}$ for $\mathbf{c}$ is any function $\mathbf{d} : \Sigma \times \mathcal{N}(\mathbf{c}) \to \mathcal{N}(\mathbf{c})$ such that $\forall \langle x,y\rangle \in E(x)$, $\pi \in P[y,z]$: $\mathbf{d}(\lambda_x(\langle x,y\rangle), \mathbf{c}(\Lambda_y(\pi))) = \mathbf{c}(\lambda_x(\langle x,y\rangle) \cdot \Lambda_y(\pi))$, where $\cdot$ is the concatenation operator. Or alternatively, $\forall a \cdot w \in \Lambda[x,z]$, $a \in \Sigma$: $\mathbf{d}(a, \mathbf{c}(w)) = \mathbf{c}(a \cdot w)$.

Definition 2 (sense of direction). *A labeled graph* (G,λ) *has* sense of direction $(\mathbf{c},\mathbf{d})$ *iff* $\mathbf{c}$ *is a weak sense of direction and* $\mathbf{d}$ *is a decoding function for* $\mathbf{c}$. *Alternatively, we shall say that* $(\mathbf{c},\mathbf{d})$ *is a sense of direction in* (G,λ).

Let $\mathcal{L}$, $\mathcal{W}$, and $\mathcal{D}$ denote the set of labeled graphs (G,λ) with local orientation, weak sense of direction, and sense of direction, respectively. The following relationship holds.

Lemma 2 (see [11, 20]). $\mathcal{D} \subset \mathcal{W} \subset \mathcal{L}$.

The proof follows from the fact that local orientation is necessary but (obviously) not sufficient for consistency [20], and the fact that there exist labeled graphs with weak sense of direction but without sense of direction [11].

2.2. Backward consistency and sense of direction. In this section we introduce the notion of backward consistency.

Definition 3. *A coding function* $\mathbf{c}$ *of* (G,λ) *is* backward consistent *if the following holds* $\forall x,y,z \in V$, $\forall \pi_1 \in P[x,z]$, $\pi_2 \in P[y,z]$*:* $\mathbf{c}(\Lambda_x(\pi_1)) = \mathbf{c}(\Lambda_y(\pi_2)) \Leftrightarrow x = y$.

In other words, sequences of labels of walks *terminating* in the same node are mapped to the same value iff they start from the same node.

Note that the difference between the consistency just defined and the traditional one is of "viewpoint." In the latter, the focus is *forward*, on the sequence of labels on walks leaving from a given node; in the former the focus is *backward,* on the sequences of labels on walks which terminate at a given node.

Definition 4 (weak backward sense of direction). *A system* (G,λ) *has* weak backward sense of direction $\mathbf{c}$ *iff* $\mathbf{c}$ *is a backward consistent coding function. Alternatively, we shall say that* $\mathbf{c}$ *is a weak backward sense of direction in* (G,λ).

Given a coding function $\mathbf{c}$, a *backward decoding function* $\mathbf{b}$ for $\mathbf{c}$ is any function $\mathbf{b} : \mathcal{N}(\mathbf{c}) \times \Sigma \to \mathcal{N}(\mathbf{c})$ such that $\forall \pi \in P[x,y]$, $\mathbf{d}(\mathrm{a},\mathbf{c}(\mathrm{w})) = \mathbf{c}(\mathrm{aw})$ $\langle y,z\rangle \in E(y)$, $\mathbf{b}(\mathbf{c}(\Lambda_x(\pi)), \lambda_y(\langle y,z\rangle)) = \mathbf{c}(\Lambda_x(\pi) \cdot \lambda_y(\langle y,z\rangle))$, where $\cdot$ is the concatenation operator. (Recall that $\mathcal{N}(\mathbf{c})$ denotes the codomain of $\mathbf{c}$.) We can now define backward sense of direction.

Definition 5 (backward sense of direction). *A system* (G,λ) *has* backward sense of direction $(\mathbf{c},\mathbf{b})$ *iff* $\mathbf{c}$ *is a weak backward sense of direction and* $\mathbf{b}$ *is a backward decoding function for* $\mathbf{c}$. *Alternatively, we shall say that* $(\mathbf{c},\mathbf{b})$ *is a backward sense of direction in* (G,λ).

We shall denote by $\mathcal{W}^-$ and $\mathcal{D}^-$ the set of labeled graphs (G,λ) with backward weak sense of direction and backward sense of direction, respectively. In the following, when no ambiguity arises, we shall omit the reference to (G,λ).

3. Backward consistency.

3.1. Absence of local orientation. The first basic difference between forward and backward consistency is striking: Local orientation is *not* implied by the definition of backward sense of direction.

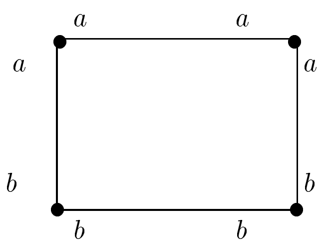

FIG. 1. $(\mathcal{D}^- - \mathcal{L}) \neq \emptyset$.

THEOREM 1 ($\mathcal{D}^- \not\subset \mathcal{L}$, $\mathcal{L} \not\subset \mathcal{D}^-$). *Local orientation is neither necessary nor sufficient for backward sense of direction.*

Proof. Consider the labeled graph (G, λ) of Figure 1. It is easy to verify that there exist both consistent backward coding and decoding functions in (G, λ). However, there is no local orientation. The fact that it is not sufficient is obvious (see, for example, the graph of Figure 4, which has local orientation but not backward sense of direction). □

Thus, in a system with backward sense of direction, nodes might not be able to distinguish among some of their incident edges. As the example in Figure 1 shows, in a system with backward sense of direction, this "blindness" can even be *complete* (i.e., it extends to all incident edges of a node) and, in the extreme case, *total* (i.e., it occurs at every node).

This phenomenon is not restricted to a few special graphs. On the contrary, *every* graph G can be labeled so as to have complete and total blindness and still have backward sense of direction.

THEOREM 2. *For any graph G there exists a labeling λ such that $\forall x \in V$, $\forall\langle x,y\rangle, \langle x,z\rangle \in E(x)$, $\lambda_x(\langle x,y\rangle) = \lambda_x(\langle x,z\rangle)$, but (G,λ) has backward sense of direction.*

Proof. Consider the labeling λ defined as follows: $\forall\langle x,y\rangle, \langle w,z\rangle \in E$, $\lambda_x(\langle x,y\rangle) = \lambda_w(\langle w,z\rangle)$ iff $x = w$ (see Figure 2). With this labeling, all the links starting from the same node x are labeled with the same label (hence, there is total and complete blindness). It is not difficult to show that the coding function $\mathbf{c}$ defined as $\mathbf{c}(a \cdot \alpha) = a$ $\forall\ a \in \Sigma, \alpha \in \Sigma^+$ is backward consistent and that the function $\mathbf{b}(\mathbf{c}(\alpha), a) = \mathbf{c}(\alpha)$ is a backward decoding for $\mathbf{c}$; hence, (G, λ) has backward sense of direction. □

Note that this is *not* the labeling used in Figure 1.

3.2. Backward local orientation. Systems with backward consistency may be without local orientation; however, they do have a property which is the "backward" analogue of local orientation. A labeling λ is a *backward local orientation* if the following holds $\forall z, \langle x,z\rangle, \langle y,z\rangle \in E(z)$: $\lambda_x(\langle x,z\rangle) \neq \lambda_y(\langle y,z\rangle)$. Let $\mathcal{L}^-$ denote the set of labeled graphs (G, λ) with backward local orientation.

First observe that backward local orientation is (obviously) *not sufficient* for backward consistency.

THEOREM 3 ($(\mathcal{L}^- - \mathcal{W}^-) \neq \emptyset$, $(\mathcal{L}^- - \mathcal{W}^-) - \mathcal{L} \neq \emptyset$). *Backward local orientation does not suffice for (either) backward consistency (or local orientation).*

Proof. Consider the labeled graph of Figure 3. Such a graph clearly has backward local orientation. By contradiction, let us assume that it has backward weak sense of direction. By definition of backward coding function we would have that, from node

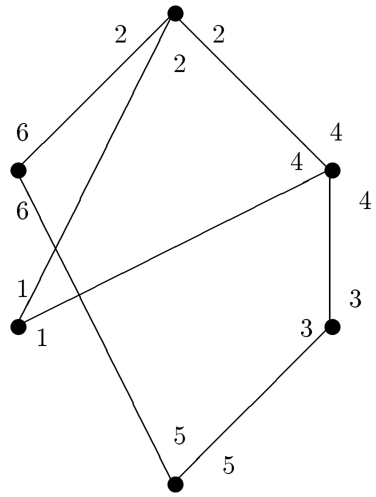

FIG. 2. *Total blindness.*

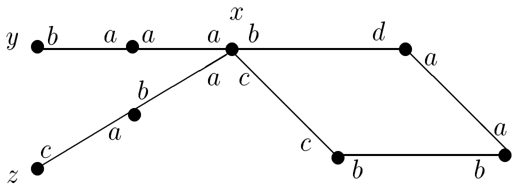

FIG. 3. $(\mathcal{L}^- - \mathcal{W}^-) - \mathcal{L} \neq \emptyset$.

x, $\mathbf{c}(b \cdot a) = \mathbf{c}(c \cdot b)$. However the labels $b \cdot a$ correspond to a path in $P[y,x]$ and $c \cdot b$ to a path in $P[z,x]$; thus, by definition of backward consistency, we must have the following: $\mathbf{c}(b \cdot a) \neq \mathbf{c}(c \cdot b)$. Since this graph does not have local orientation, this actually proves that $(\mathcal{L}^- - \mathcal{W}^-) - \mathcal{L} \neq \emptyset$. $\square$

However, in a system with weak backward sense of direction, there is always backward local orientation.

THEOREM 4 ($\mathcal{W}^- \subset \mathcal{L}^-$). *If (G,λ) has weak backward sense of direction, then λ has backward local orientation.*

Proof. Let (G,λ) be a labeled graph without backward local orientation and with backward consistency. Then there exist two edges $\langle x,z\rangle$ and $\langle y,z\rangle$ such that $\lambda_x(\langle x,z\rangle) = \lambda_y(\langle y,z\rangle)$. But this contradicts the existence of a backward coding function $\mathbf{c}$, as by definition $\mathbf{c}(\lambda_x(\langle x,z\rangle)) = \mathbf{c}(\lambda_y(\langle y,z\rangle))$. $\square$

Summarizing, (backward) local orientation is necessary but not sufficient for (backward) consistency. Thus, $\mathcal{W}^- \subset \mathcal{L}^-$.

Two questions naturally arise: Is the simultaneous presence of both local orientation and backward local orientation sufficient for consistency? Is it sufficient for backward consistency? The answer to these questions is negative, as we prove the stronger result that there exist labeled graphs having both forms of local orientation but without either form of consistency.

THEOREM 5 ($(\mathcal{L} \cap \mathcal{L}^-) - (\mathcal{W} \cup \mathcal{W}^-) \neq \emptyset$). *The simultaneous presence of local ori-*

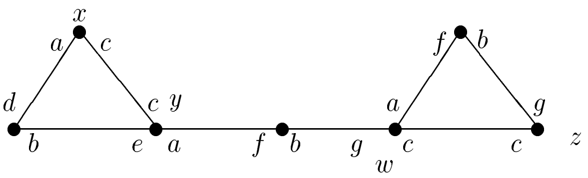

FIG. 4. $(\mathcal{L} \cap \mathcal{L}^-) - (\mathcal{W} \cup \mathcal{W}^-) \neq \emptyset$.

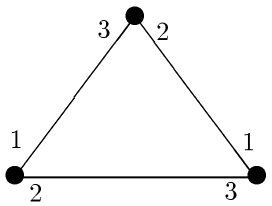

FIG. 5. $\mathcal{D} - \mathcal{L}^- \neq \emptyset$.

entation and backward local orientation does not imply either weak sense of direction or backward weak sense of direction.

Proof. Consider the labeled graph (G, λ) shown in Figure 4. The labeling clearly has both local orientation and backward local orientation. By contradiction, assume (G, λ) has weak sense of direction $\mathbf{c}$. By definition of consistent coding, at node x it must be $\mathbf{c}(a \cdot b) = \mathbf{c}(c)$; however, at node y it must be $\mathbf{c}(a \cdot b) \neq \mathbf{c}(c)$, yielding a contradiction. Assume now, again by contradiction, that (G, λ) has backward weak sense of direction $\mathbf{c}$. By definition of backward consistent coding, at node z it must be $\mathbf{c}(a \cdot b) = \mathbf{c}(c)$; however, at node w it must be $\mathbf{c}(a \cdot b) \neq \mathbf{c}(c)$, which is a contradiction. □

3.3. Orthogonality. In this section we will show that the concept of backward consistency is *orthogonal* to that of sense of direction; in particular, even the presence of sense of direction is not sufficient to guarantee the existence of a backward consistency.

THEOREM 6 ($\mathcal{D} \not\subset \mathcal{L}^-$, $\mathcal{L}^- \not\subset \mathcal{D}$). *Sense of direction is neither necessary nor sufficient for the existence of backward local orientation.*

Proof. Consider any graph with more than two nodes labeled with a *neighboring* labeling (e.g., see Figure 5); all such labeled graphs have sense of direction [20]: the coding function $\mathbf{c}$ is $\mathbf{c}(\alpha \cdot a) = a \; \forall a \in \Sigma, \alpha \in \Sigma^+$, and the decoding function is $\mathbf{d}(a, \mathbf{c}(\alpha)) = \mathbf{c}(\alpha)$. On the other hand, (G, λ) does not have backward local orientation. In other words, sense of direction is not sufficient for the existence of backward local orientation. The fact that it is also not necessary follows from Theorem 5. □

In other words, sense of direction cannot guarantee backward consistency because it cannot even guarantee backward local orientation. What if there is backward local orientation in addition to sense of direction? The answer is still negative, as shown in the following theorem.

THEOREM 7 ($(\mathcal{D} \cap \mathcal{L}^-) \not\subset \mathcal{W}^-$, $(\mathcal{D} \cap \mathcal{L}^-) \not\supset \mathcal{W}^-$). *Simultaneous presence of sense of direction and backward local orientation is neither necessary nor sufficient for the existence of a backward consistent coding function.*

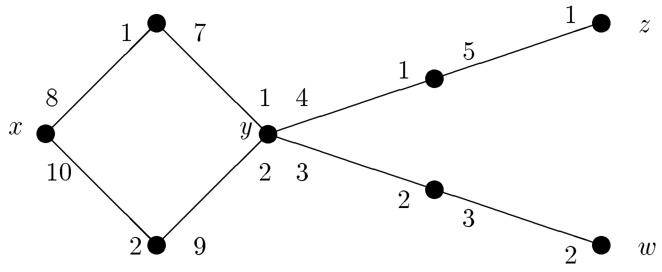

FIG. 6. $(\mathcal{D} \cap \mathcal{L}^-) \not\supset \mathcal{W}^-$.

Proof. $\not\subset$. To prove that it is not sufficient, consider the labeled graph (G, λ) of Figure 6. It is easy to verify that there exist both a consistent coding function and decoding function in (G, λ). By contradiction, assume that there exists a backward consistent coding function $\mathbf{c}$ in (G, λ). Then, by definition of backward consistency at node x we have that $\mathbf{c}(1 \cdot 1) = \mathbf{c}(2 \cdot 2)$. However, the sequence of labels $1 \cdot 1$ corresponds to a path in $P[z, y]$ and the sequence $2 \cdot 2$ to a path in $P[w, y]$; thus, by definition of backward consistency, $\mathbf{c}(1 \cdot 1) \neq \mathbf{c}(2 \cdot 2)$, yielding the contradiction.

$\not\supset$. Now we will prove that it is not necessary. By Theorem 4, backward consistency implies backward local orientation. However, from Theorem 6, it follows that backward consistency does not imply local orientation and, thus, does not imply sense of direction. This means that the simultaneous presence of sense of direction and backward local orientation is not necessary for backward consistency. ∎

The results of this section indicate that the presence of some additional property is necessary for one type of consistency to imply the other. We will show in the next section that *edge-symmetry* is one such property.

4. Symmetry and backward consistency. The results of the previous section indicate that the presence of some additional property is necessary for one type of consistency to imply the other. We will show that *edge-symmetry* is one such property.

Recall that a labeling λ is *symmetric* if there exists a bijection $\psi : \Sigma \to \Sigma$ such that for each $\langle x, y\rangle \in E$, $\lambda_y(\langle y, x\rangle) = \psi(\lambda_x(\langle x, y\rangle))$; ψ will be called the edge-symmetry function, and $\Psi : \Sigma^+ \to \Sigma^+$ is its extension to strings. Let $\mathcal{ES}$ denote the set of labeled graphs (G, λ) with λ symmetric.

Notice that all common labelings (e.g., "dimensional" in hypercubes, "compass" in meshes and tori, "left-right" in rings, "distance" in chordal rings, etc.) are symmetric.

4.1. Edge-symmetry and backward consistency. We will show that, while generally not true (as shown by Theorem 1), in systems with edge-symmetry, local orientation is necessary for backward consistency. In fact, we shall show that in systems with edge-symmetry, $\mathcal{L} = \mathcal{L}^-$.

THEOREM 8 ($\mathcal{ES} \cap \mathcal{L} = \mathcal{ES} \cap \mathcal{L}^-$). *Let (G, λ) be a system with edge-symmetry; then there is local orientation iff there is backward local orientation.*

Proof. Let (G, λ) be a system with edge-symmetry. $\subset$. By contradiction, suppose there is local orientation but no backward local orientation. Then there must exist two distinct edges $\langle x, z\rangle$, $\langle y, z\rangle$ such that $\lambda_x(\langle x, z\rangle) = \lambda_y(\langle y, z\rangle)$. However, by local orientation, we have that $\lambda_z(\langle z, x\rangle) \neq \lambda_z(\langle z, y\rangle)$, which contradicts the fact that there is edge-symmetry.

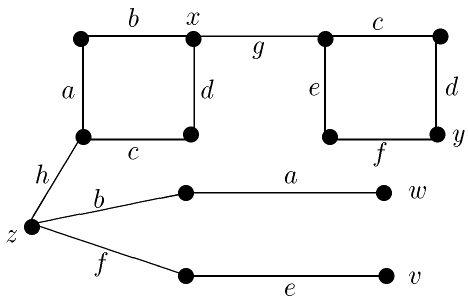

FIG. 7. $\mathcal{ES} \cap \mathcal{L} \cap \mathcal{L}^- \not\subset \mathcal{W}^-$.

$\supset$. By contradiction, suppose now that there is backward local orientation but no local orientation. Then there must exist two distinct edges $\langle x, y\rangle$, $\langle x, z\rangle$ such that $\lambda_x(\langle x, y\rangle) = \lambda_x(\langle x, z\rangle)$. However, by backward local orientation we must have $\lambda_y(\langle y, x\rangle) \neq \lambda_z(\langle z, y\rangle)$, contradicting edge-symmetry of the labeling. $\square$

Thus, a system with edge-symmetry has either both types of local orientation or none; still, presence of both types and edge-symmetry are not sufficient for backward consistency.

THEOREM 9 ($\mathcal{ES} \cap \mathcal{L} \cap \mathcal{L}^- \not\subset \mathcal{W}^-$). *The simultaneous presence of local orientation and edge-symmetry is not sufficient for the existence of backward consistency.*

Proof. Consider the labeled graph of Figure 7. The labeling is a coloring, i.e., the edge-symmetry function is the identity function; moreover, it is a local orientation. By contradiction, assume that there exists a backward consistent coding function $\mathbf{c}$. By definition of backward consistency, at node x we have that $\mathbf{c}(a \cdot b) = \mathbf{c}(c \cdot d)$ and at node y, that $\mathbf{c}(c \cdot d) = \mathbf{c}(e \cdot f)$, which implies $\mathbf{c}(a \cdot b) = \mathbf{c}(e \cdot f)$. However, the sequence of labels $b \cdot a$ corresponds to a path in $P[w, z]$, and $f \cdot e$ corresponds to a path in $P[v, z]$; thus we have that $\mathbf{c}(a \cdot b) \neq \mathbf{c}(e \cdot f)$, which is a contradiction. $\square$

We will now show that edge-symmetry in a system with (weak) sense of direction suffices for the system to have (weak) backward sense of direction.

LEMMA 3 (see [20]). $\forall \pi \in P[x, y]$, $\Lambda_x(\pi) = \Psi(\Lambda_y(\pi^R))$, *where* $\pi^R \in P[y, x]$ *is the reverse of walk* π.

THEOREM 10 ($\mathcal{ES} \cap \mathcal{W} \subset \mathcal{W}^-$, $\mathcal{ES} \cap \mathcal{D} \subset \mathcal{D}^-$). *Let* (G, λ) *be a system with edge-symmetry. If* (G, λ) *has weak sense of direction* $\mathbf{c}$, *then there exists a weak backward sense of direction* $\mathbf{c}_b$. *Furthermore, if* $\mathbf{c}$ *has a consistent decoding, there exists a consistent backward decoding of* $\mathbf{c}_b$.

Proof. Let (G, λ) have edge-symmetry and weak sense of direction $\mathbf{c}$. Let $\mathbf{c}_b$ be the coding function defined as follows: $\forall \pi \in P[x]$, $\mathbf{c}_b(\alpha) = \mathbf{c}(\Psi(\alpha))$, where $\alpha = \Lambda_x(\pi)$. We will now show that $\mathbf{c}_b$ is backward consistent.

Let $\pi_1 \in P[x, z]$, $\pi_2 \in P[y, z]$, $\alpha_1 = \Lambda_x(\pi_1)$, and $\alpha_2 = \Lambda_y(\pi_2)$. By Lemma 3, we have that $\alpha_1 = \Psi(\Lambda_z(\pi_1^R))$ and $\alpha_2 = \Psi(\Lambda_z(\pi_2^R))$. By definition of the consistent coding function, we have that $\mathbf{c}(\Psi(\Lambda_z(\pi_1^R))) = \mathbf{c}(\Psi(\Lambda_z(\pi_2^R)))$ iff $x = y$. By definition of $\mathbf{c}_b$, it follows that $\mathbf{c}_b(\Lambda_x(\pi_1)) = \mathbf{c}_b(\Lambda_y(\pi_2))$ iff $y = z$; that is, $\mathbf{c}_b$ is backward consistent.

Now let (G, λ) have edge-symmetry and sense of direction $(\mathbf{c}, \mathbf{d})$. Let $\mathbf{b}$ be the function defined as follows: $\forall \pi \in P[x, y]$, $\langle y, z\rangle \in E(y)$, $\mathbf{b}(\mathbf{c}_b(\alpha), \lambda_y(\langle y, z\rangle)) = \mathbf{d}(\psi(\lambda_y(\langle y, z\rangle)), \mathbf{c}_b(\alpha))$, where $\alpha = \Lambda_x(\pi)$. We will now show that $\mathbf{b}$ is a backward

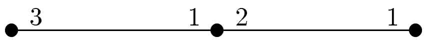

FIG. 8. $\mathcal{W} \cap \mathcal{W}^- \not\subset \mathcal{ES}$.

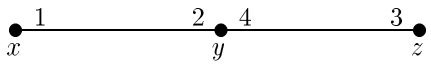

FIG. 9. *Proof of Theorem* 13.

decoding. By definition of $\mathbf{c}_b$ we have that $\mathbf{d}(\psi(\lambda_y(\langle y,z\rangle)), \mathbf{c}_b(\alpha)) = \mathbf{d}(\psi(\lambda_y(\langle y,z\rangle)), \mathbf{c}(\Psi(\alpha)))$. By definition of the decoding function we have that $\mathbf{d}(\psi(\lambda_y(\langle y,z\rangle)), \mathbf{c}(\Psi(\alpha))) = \mathbf{c}(\psi(\lambda_y(\langle y,z\rangle))\cdot\Psi(\alpha))$. But, by definition of $\mathbf{c}$, we have that $\mathbf{c}(\psi(\lambda_y(\langle y,z\rangle))\cdot\Psi(\alpha)) = \mathbf{c}_b(\alpha\cdot\lambda_y(\langle y,z\rangle))$. Thus, it follows that $\mathbf{b}(\mathbf{c}_b(\alpha), \lambda_y(\langle y,z\rangle)) = \mathbf{c}_b(\alpha\cdot\lambda_y(\langle y,z\rangle))$, proving that $\mathbf{b}$ is a backward decoding of $\mathbf{c}_b$. □

Conversely, edge-symmetry in a system with (weak) backward sense of direction suffices for the system to have (weak) sense of direction.

THEOREM 11 ($(\mathcal{ES} \cap \mathcal{W}^-) \subset \mathcal{W}$, $(\mathcal{ES} \cap \mathcal{D}^-) \subset \mathcal{D}$). *Let (G,λ) be a system with edge-symmetry. If (G,λ) has weak backward sense of direction $\mathbf{c}_b$, then there exists a weak sense of direction $\mathbf{c}$. Furthermore, if $\mathbf{c}_b$ has a consistent backward decoding, there exists a consistent decoding of $\mathbf{c}$.*

From Theorems 10 and 11, it follows that systems with edge-symmetry have either both types of consistency or none. In other words, in systems with edge-symmetry, $\mathcal{W} = \mathcal{W}^-$ and $\mathcal{D} = \mathcal{D}^-$.

An immediate question is whether systems with edge-symmetry are the only ones with such a property.

THEOREM 12 ($\mathcal{W} \cap \mathcal{W}^- \not\subset \mathcal{ES}$). *Edge-symmetry is not necessary for a system to have both forward and backward consistency.*

Proof. Consider the graph of Figure 8. This graph does not have edge-symmetry; however, it is easy to verify that there exist both a forward consistent coding function $\mathbf{c}$, and a backward consistent coding function $\mathbf{c}_b$. □

4.2. Edge-symmetry and biconsistency. In the previous section we have seen that all systems with edge-symmetry have either both types of consistency or neither. For those that do have them, the corresponding coding functions are in general different.

The question we ask now is under what circumstances a single coding function suffices, that is, when a coding function is both forward and backward consistent. We shall call any such function *biconsistent.* Let $\mathcal{W}^\pm$ and $\mathcal{D}^\pm$ denote the set of labeled graphs (G,λ) with biconsistent weak sense of direction and biconsistent sense of direction, respectively. The first result is the following.

THEOREM 13. *Edge-symmetry is not sufficient for a (backward) consistent coding function to be biconsistent.*

Proof. Consider the labeled graph in Figure 9. This labeling has local orientation. Consider any coding function such that $\mathbf{c}(1) = \mathbf{c}(3)$, $\mathbf{c}(2) \neq \mathbf{c}(4)$, $\mathbf{c}(1,2) = \mathbf{c}(1432)$, $\mathbf{c}(3,4) = \mathbf{c}(3214)$, $\mathbf{c}(2,1) = \mathbf{c}(43)$, and $\forall \alpha,\beta \in \{1,2,3,4\}^*$, $\mathbf{c}(\alpha 12\beta) = \mathbf{c}(\alpha 21\beta) = \mathbf{c}(\alpha 34\beta) = \mathbf{c}(\alpha 43\beta) = \mathbf{c}(\alpha\beta)$.

It is easy to verify that such a coding function is consistent. However, $\mathbf{c}$ is not biconsistent. In fact, $\mathbf{c}(1) = \mathbf{c}(3)$, which violates the backward consistency in y. □

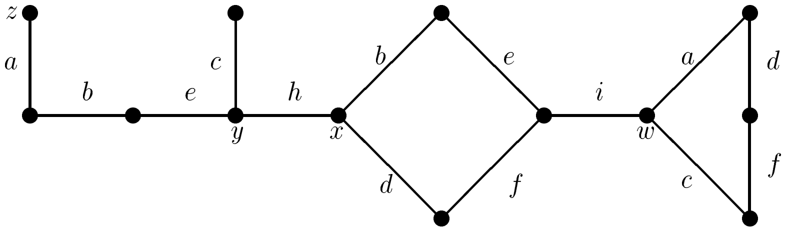

FIG. 10. $\mathcal{ES} \cap \mathcal{D}^- \not\subset \mathcal{ES} \cap \mathcal{D}$.

Even if not all the coding functions in a symmetric labeling are biconsistent, there exists at least one biconsistent coding function.

THEOREM 14 ($\mathcal{ES} \cap \mathcal{W} = \mathcal{ES} \cap \mathcal{W}^- = \mathcal{ES} \cap \mathcal{W}^{\pm}$). *In a system (G, λ) with edge-symmetry and (backward) weak sense of direction there exists also a biconsistent weak sense of direction.*

Proof. Let $\overline{\mathbf{c}}$ be a weak sense of direction in (G, λ) with edge-symmetry; we will show how to construct a weak sense of direction $\mathbf{c}$ which is biconsistent. We define the equivalence relation R between strings on the labeling alphabet in the following way: $\forall \alpha, \beta \in \Sigma^+ : \alpha R \beta \Leftrightarrow \exists x, y \in V, \pi_1, \pi_2 \in P[x, y] : \Lambda_x(\pi_1) = \alpha \wedge \Lambda_x(\pi_2) = \beta$. Let R^+ be the transitive closure of R. Let $c(\alpha) = [\alpha]_{R^+}$ (i.e., the equivalence class of the string α of the equivalence R^+).

CLAIM 1.

(i) *For all weak sense of direction $\mathbf{c}', \alpha, \beta \in \Sigma^+ : \mathbf{c}(\alpha) = \mathbf{c}(\beta) \Rightarrow \mathbf{c}'(\alpha) = \mathbf{c}'(\beta)$.*

(ii) $\forall \alpha, \beta : \mathbf{c}(\alpha) = \mathbf{c}(\beta) \Leftrightarrow \mathbf{c}(\psi(\alpha)) = \mathbf{c}(\psi(\beta))$.

(iii) $[\psi(\alpha)]_{R^+} = \psi([\alpha]_{R^+})$.

We now prove that $\mathbf{c}$ is a consistent coding function. Let $x, y, z \in V, \pi_1 \in P[x, y] \pi_2 \in P[x, z] : \Lambda_x(\pi_1) = \alpha \wedge \Lambda_x(\pi_2) = \beta$; then we have to prove that $\mathbf{c}(\alpha) = \mathbf{c}(\beta)$ iff $y = z$. By Claim 1(i), $\mathbf{c}(\alpha) = \mathbf{c}(\beta)$ implies $\overline{\mathbf{c}}(\alpha) = \overline{\mathbf{c}}(\beta)$, and by consistence of $\overline{\mathbf{c}}$ it follows that $y = z$. If $y = z$, then by definition of R it holds that $\alpha R \beta$, and then $[\alpha]_{R^+} = [\beta]_{R^+}$, that is, $\mathbf{c}(\alpha) = \mathbf{c}(\beta)$.

We now prove that $\mathbf{c}$ is a backward consistent coding function. By contradiction suppose that $\mathbf{c}$ is not consistent. Then there exists x, y, z and $\pi_1 \in P[x, z], \pi_2 \in P[y, z]$ for which $\mathbf{c}(\Lambda_x(\pi_1)) = \mathbf{c}(\Lambda(\pi_2)) \Leftrightarrow x = y$ is false. By Claim 1(i), this implies that there exists $x, y, z, \pi_1' \in P[z, x], \pi_2' \in P[z, y] : \Lambda_z(\pi_1') = \psi(\alpha)$, $\Lambda_z(\pi_2') = \psi(\beta)$, and $\mathbf{c}(\psi(\alpha)) = \mathbf{c}(\psi(\beta)) \Leftrightarrow x = y$ is false, contradicting the forward consistency of $\mathbf{c}$. ∎

The presence of biconsistency guaranteed by the above theorem is automatically true only for weak sense of direction. This property, unfortunately, does not extend to sense of direction, as the following theorem shows.

THEOREM 15 ($\mathcal{ES} \cap \mathcal{D} \not\subset \mathcal{W}^{\pm}$). *Sense of direction in a system with edge-symmetry is not sufficient for the existence of a biconsistent coding function.*

Proof. Consider the labeled graph of Figure 10. It is easy to verify that there exists a sense of direction for such a system. Let $(\mathbf{c}, \mathbf{d})$ be a sense of direction and suppose, by contradiction, that $\mathbf{c}$ is backward consistent.

By definition of the coding function at node x we have that $\mathbf{c}(df) = \mathbf{c}(be)$; it follows that $\mathbf{d}(a, \mathbf{c}(be)) = \mathbf{d}(a, \mathbf{c}(df)) = \mathbf{c}(c)$, which means that $\mathbf{c}(abe) = \mathbf{c}(c)$, contradicting the hypothesis of backward consistency at node y. ∎

As a consequence of the previous theorem we have that sense of direction in a system with edge-symmetry is not sufficient for the existence of a backward sense of direction (i.e., $\mathcal{ES} \cap \mathcal{D} \not\subset \mathcal{W}^{\pm}$).

Now we will study a sufficient property of a consistent function to be biconsistent. A weak sense of direction $\mathbf{c}$ has *name-symmetry* iff there exists a function $\mu : \mathcal{N}(\mathbf{c}) \rightarrow \mathcal{N}(\mathbf{c})$ such that $\forall \pi \in P[x,y]$, $\mu(\mathbf{c}(\Lambda_x(\pi))) = \mathbf{c}(\Lambda_y(\pi^R))$. A useful property is the following [22] lemma.

LEMMA 4 (see [22]). *Let λ be a symmetric labeling and let ψ be the corresponding function. A consistent coding function $\mathbf{c}$ has name-symmetry iff $\forall \pi_1 \in P[s,t], \pi_2 \in P[w,z]$, $\mathbf{c}(\Lambda_s(\pi_1)) = \mathbf{c}(\Lambda_w(\pi_2)) \Rightarrow \mathbf{c}(\Psi(\Lambda_s(\pi_1))) = \mathbf{c}(\Psi(\Lambda_w(\pi_2)))$.*

THEOREM 16. *In a system (G,λ) with edge-symmetry, any weak sense of direction with name-symmetry is also weak backward sense of direction.*

Proof. Let $\mathbf{c}$ be a weak sense of direction in (G,λ) with edge- and name-symmetry, and let ψ be the edge-symmetry function. By contradiction, suppose that c is not a consistent backward coding function; that is, suppose $\exists x,y,z,\ \pi_1 \in P[x,z]$, $\pi_2 \in P[y,z]$ with $x \neq y$ such that $\mathbf{c}(\Lambda_x(\pi_1)) = \mathbf{c}(\Lambda_y(\pi_2))$. Since there is name-symmetry, by Lemma 4, we have that $\mathbf{c}(\Psi(\Lambda_x(\pi_1))) = \mathbf{c}(\Psi(\Lambda_y(\pi_2)))$. But $\Lambda_z(\overline{\pi}_1) = \Psi(\Lambda_x(\pi_1))$, $\Lambda_z(\overline{\pi}_2) = \Psi(\Lambda_y(\pi_2))$ and $\overline{\pi}_1 \in P[z,x]$, $\overline{\pi}_2 \in P[z,y]$, which contradicts the consistency of $\mathbf{c}$ in z. ◻

In other words, in systems with edge-symmetry, any consistent coding function with name-symmetry is also backward consistent. We will now show that if any such coding function is decodable, then it is also backward decodable; that is, if there is sense of direction, there is also backward sense of direction with exactly the same coding function.

LEMMA 5. *Let $\mathbf{c}$ be a coding function with name-symmetry, and let μ be the name-symmetry function. For any $\alpha \in \Sigma^+$ corresponding to a path, $\mu(\mu(\mathbf{c}(\alpha))) = \mathbf{c}(\alpha)$.*

Proof. By definition of the name-symmetry function, we have $\mu(\mu(\mathbf{c}(\alpha))) = \mu(\mathbf{c}(\Psi(\alpha))) = \mathbf{c}(\Psi(\Psi(\alpha))) = \mathbf{c}(\alpha)$. ◻

THEOREM 17. *Let $(\mathbf{c},\mathbf{d})$ be a sense of direction in a system (G,λ) with edge- and name-symmetry. Then there exists a backward decoding $\mathbf{b}$ of $\mathbf{c}$; that is, $(\mathbf{c},\mathbf{b})$ is a backward sense of direction in (G,λ).*

Proof. Let ψ and μ be, respectively, the edge- and name-symmetry functions. Let $x,y \in V$, $\pi \in P[x,y]$, and $\omega = \Lambda_x(\pi)$. By definition of the name-symmetry function we have that $\mathbf{c}(\Psi(\omega)) = \mu(\mathbf{c}(\omega))$; it follows that $\mu(\mathbf{c}(\Psi(\omega))) = \mu(\mu(\mathbf{c}(\omega)))$. By Lemma 5, we have that

$$\mu(\mathbf{c}(\Psi(\omega))) = \mathbf{c}(\omega). \tag{4.1}$$

Consider now the following backward decoding function: $\forall \pi \in P[x,y]$, $\omega = \Lambda_x(\pi)$ and for $\langle y,z\rangle \in E(y) \wedge \lambda_y(\langle y,z\rangle) = a$, $\mathbf{b}(\mathbf{c}(\omega),a) = \mu(\mathbf{d}(\psi(a),\mu(\mathbf{c}(\omega))))$.

By definition of the name-symmetry function, it follows that $\mu(\mathbf{d}(\psi(a),\mu(\mathbf{c}(\omega))) = \mu(\mathbf{d}(\psi(a),\mathbf{c}(\Psi(\omega)))) = \mu(\mathbf{c}(\Psi(w \cdot a)))$.

By (4.1), $\mu(\mathbf{c}(\Psi(w \cdot a))) = \mathbf{c}(\omega \cdot a)$. It follows that $\mathbf{b}(\mathbf{c}(\omega),a) = \mathbf{c}(\omega \cdot a)$, and thus $\mathbf{b}$ is consistent. ◻

5. Consistency landscape. The results of the previous sections can be seen, and have sometimes been stated, in terms of the "consistency landscape" (see Figure 11), i.e., of the relationship among the sets $\mathcal{L}, \mathcal{W}, \mathcal{D}, \mathcal{L}^-, \mathcal{W}^-, \mathcal{D}^-$.

In this section, we continue the analysis of the "consistency landscape."

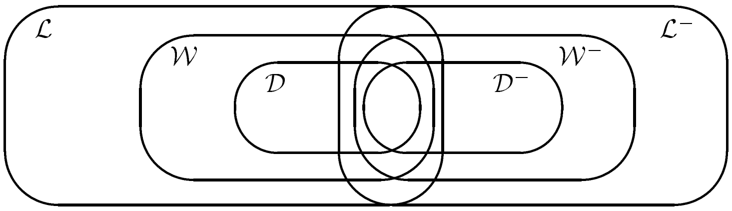

FIG. 11. *The consistency landscape.*

5.1. Transformations and constructions. In this section, we introduce two useful operations on labeled graphs. In the end, we shall derive general results on the structure of the consistency landscape.

The first operation, called "doubling," allows us to transform a labeling into a *symmetric* one. Hence, it allows us to transform a system with only one type of consistency into one which has both.

Given (G, λ), the *doubling* of λ is the mapping $\lambda^2_x((x, y))$, defined as follows: for all edges $\langle x, y\rangle$, $\lambda^2_x(\langle x, y\rangle) = (\lambda_x(\langle x, y\rangle), \lambda_y(\langle y, x\rangle))$. Its extension from edges to walks will be denoted by Λ^2.

The most important property of double labeling is that if (G, λ) has either form of consistency, then (G, λ^2) has both.

Given two strings of equal length $\alpha = a_0a_1 \dots a_k \in \Sigma^+$ and $\beta = b_0b_1 \dots b_k \in \Sigma^+$, let $\alpha \oplus \beta = (a_0, b_0)(a_1, b_1) \dots (a_k, b_k) \in (\Sigma^2)^+$ denote their product. Given a string $\alpha = a_0a_1 \dots a_k \in \Sigma^+$, let $\alpha^R = a_ka_{k-1} \dots a_0$ denote the reverse string.

THEOREM 18 (double labeling). *If (G, λ) has either (weak) sense of direction or (weak) backward sense of direction, then (G, λ^2) has both (weak) sense of direction and (weak) backward sense of direction.*

Proof. Let $\mathbf{c}$ be a coding function in (G, λ). Consider the coding function $\mathbf{c}_2$ defined as follows: $\forall\ \alpha \oplus \beta \in (\Sigma^2)^+$, $\mathbf{c}_2(\alpha \oplus \beta) = \mathbf{c}(\alpha)$. Clearly, $\mathbf{c}_2$ is (resp., backward) consistent in (G, λ^2) iff $\mathbf{c}$ is (resp., backward) consistent in (G, λ).

Let $\mathbf{c}$, and hence $\mathbf{c}_2$, be (resp., backward) consistent. Given a function $\mathbf{d} : \Sigma \times \mathcal{N}(\mathbf{c}) \to \mathcal{N}(\mathbf{c})$, define the function $\mathbf{d}_2$ as follows: $\forall\ (a, b) \in \Sigma^2$ and $\alpha \oplus \beta \in (\Sigma^2)^+$, $\mathbf{d}_2((a, b), \mathbf{c}_2(\alpha \oplus \beta)) = \mathbf{d}(a, \mathbf{c}(\alpha))$. Clearly, $\mathbf{d}_2$ is a decoding of $\mathbf{c}_2$ iff $\mathbf{d}$ is a decoding of $\mathbf{c}$. Similarly, given $\mathbf{b} : \mathcal{N}(\mathbf{c}) \times \Sigma \to \mathcal{N}(\mathbf{c})$, the function $\mathbf{b}_2(\mathbf{c}_2(\alpha \oplus \beta), (a, b)) = \mathbf{d}(a, \mathbf{c}(\alpha))$ is a backward decoding of c_2 iff $\mathbf{b}$ is a backward decoding of $\mathbf{c}$.

In other words, (weak) sense of direction in (G, λ) implies (weak) sense of direction in (G, λ^2), and (weak) backward sense of direction in (G, λ) implies (weak) backward sense of direction in (G, λ^2).

Finally observe that λ^2 is symmetric; in fact, for every $\langle x, y\rangle \in E$, $\lambda^2_x(\langle x, y\rangle) = \lambda^2_y(\langle y, x\rangle)^R$. Thus, by Theorems 10 and 11, if (G, λ^2) has one type of consistency, it has both. $\square$

The operation of doubling is important in that it allows us to transform a labeling into a *symmetric* one, and thus to extend the existing consistency to include also the other type.

The proof of Theorem 18 shows also how to employ the (resp., backward) coding and decoding of the original system to construct the analogous ones of the new system. However, it does not give any indication on the nature of the coding and decoding of the opposite type. As we will see later, such a nature is useful also for the other transformation to be discussed in this section.

The purpose of the following lemmas is to show how to construct in the new systems the coding and decoding of the opposite type.

LEMMA 6. *Let* $\mathbf{c}$ *be a weak sense of direction in* (G,λ)*; then* $\mathbf{c}_b(\alpha \oplus \beta) = \mathbf{c}(\beta^R)$ *is a weak backward sense of direction in* (G,λ^2)*. Furthermore, if* $\mathbf{c}$ *has a decoding* $\mathbf{d}$*, then* $\mathbf{b}(\mathbf{c}_b(\alpha \oplus \beta),(a,b)) = \mathbf{d}(b,\mathbf{c}(\beta^R))$ *is a backward decoding of* $\mathbf{c}_b$*.*

Proof. Let $x,y,z \in V$, $\pi_1 \in P[x,z]$, $\pi_2 \in P[y,z]$. Let $\Lambda^2_x(\pi_1) = (a_0,a'_0)\ (a_1,a'_1) \dots (a_h,a'_h)$, and $\Lambda^2_y(\pi_2) = (b_0,b'_0)(b_1,b'_1)\dots(b_k,b'_k)$.

Let $x = y$. By definition of $\mathbf{c}_b$ we have that $\mathbf{c}_b(\Lambda^2_x(\pi_1)) = \mathbf{c}(a'_h \dots a'_0) = \mathbf{c}(\Lambda_z(\overline{\pi_1})$. Since $x = y$, by definition of the consistency of $\mathbf{c}$ we have that $\mathbf{c}(\Lambda_z({\pi_1}^R)) = \mathbf{c}(\Lambda_z({\pi_2}^R)) = \mathbf{c}(b'_k \dots b'_0)$. Since $\mathbf{c}(b'_k \dots b'_0) = \mathbf{c}_b(\Lambda^2_y(\pi_2))$, it follows that $\mathbf{c}_b(\Lambda^2_x(\pi_1)) = \mathbf{c}_b(\Lambda^2_y(\pi_2))$.

Let $x \neq y$. By definition of $\mathbf{c}_b$ we have that $\mathbf{c}_b(\Lambda^2_x(\pi_1)) = \mathbf{c}(a'_h \dots a'_0)$. Since $x \neq y$, by definition of the consistency of $\mathbf{c}$ we have that $\mathbf{c}(\Lambda_z({\pi_1}^R)) \neq \mathbf{c}(\Lambda_z({\pi_2}^R))$. But $\Lambda_z({\pi_1}^R) = (a'_h \dots a'_0)$, and $\Lambda_z({\pi_2}^R) = b'_k \dots b'_0$. Since $\mathbf{c}(b'_k \dots b'_0) = \mathbf{c}_b(\Lambda^2_y(\pi_2))$, it follows that $\mathbf{c}_b(\Lambda^2_x(\pi_1)) \neq \mathbf{c}_b(\Lambda_y(\pi_2))$.

We now show that the corresponding backward consistent decoding function is the following: $\forall \pi \in P[x,y]$, $\forall \langle y,z\rangle \in E(y)$, let $\Lambda^2_x(\pi) = (\alpha \oplus \beta))$ and $\lambda^2_y(\langle y,z\rangle) = (a,b)$; then $\mathbf{b}(\mathbf{c}_b(\alpha \oplus \beta),(a,b)) = \mathbf{d}(b,\mathbf{c}(\beta^R)$.

By definition of the consistent decoding function, $\mathbf{d}(b,\mathbf{c}(\beta^R) = \mathbf{c}(b \cdot \beta^R)$. By definition of $\mathbf{c}_b$ we have that $\mathbf{c}(b \cdot \beta^R) = \mathbf{c}_b((\alpha \oplus \beta)\cdot(a,b))$. Thus, it follows that $\mathbf{b}$ is consistent, i.e., $\mathbf{b}(\mathbf{c}_b(\alpha \oplus \beta),(a,b)) = \mathbf{c}_b((\alpha \oplus \beta)\cdot(a,b))$. ◻

Similarly, we have the following lemma.

LEMMA 7. *Let* $\mathbf{c}$ *be a weak backward sense of direction in* (G,λ)*. Then* $\mathbf{c}_f(\alpha \oplus \beta) = \mathbf{c}(\beta^R)$ *is a weak sense of direction in* (G,λ^2)*. Furthermore, if* $\mathbf{c}$ *has a backward decoding* $\mathbf{b}$*, then* $\mathbf{d}((a,b),\mathbf{c}_f(\alpha \oplus \beta)) = \mathbf{b}(\mathbf{c}(\beta^R),b)$ *is a decoding of* $\mathbf{c}_f$*.*

The second transformation is "reversal." Given a labeling λ of a graph G, the *reverse labeling* $\tilde{\lambda}$ of G is obtained in the following way: $\forall \langle x,y\rangle \in E$, $\tilde{\lambda}_x(\langle x,y\rangle) = \lambda_y(\langle y,x\rangle)$.

The reversal operation is related to doubling as follows, where $\mathbf{c}_b$, $\mathbf{c}_f$, $\mathbf{b}$, and $\mathbf{d}$ are as defined in Lemmas 6 and 7.

LEMMA 8. *Let* $\mathbf{c}$ *be a weak sense of direction in* (G,λ)*; then* $\mathbf{c}_b$ *is a weak backward sense of direction in* $(G,\tilde{\lambda})$*. Furthermore, if* $\mathbf{c}$ *has a decoding* $\mathbf{d}$*, then* $(\mathbf{c}_b,\mathbf{b})$ *is backward sense of direction in* $(G,\tilde{\lambda})$*.*

LEMMA 9. *Let* $\mathbf{c}$ *be a weak backward sense of direction in* (G,λ)*; then* $\mathbf{c}_f$ *is a weak sense of direction in* $(G,\tilde{\lambda})$*. Furthermore, if* $\mathbf{c}$ *has a backward decoding* $\mathbf{b}$*, then* $(\mathbf{c}_f,\mathbf{d})$ *is a sense of direction in* $(G,\tilde{\lambda})$*.*

Proof. It directly follows from Theorem 18 that the following coding function is consistent: $\forall \pi \in P[x_0]$, $\pi = \langle x_0,x_1\rangle \dots \langle x_{m-1},x_m\rangle$: $\mathbf{c}_b(\tilde{\Lambda}_x(\pi)) = \mathbf{c}(\tilde{\lambda}_{x_{m-1}}(\langle x_{m-1},x_m\rangle) \dots \tilde{\lambda}_{x_0}(\langle x_0,x_1\rangle))$. The corresponding backward decoding function is the following: $\forall \pi \in P[x_0]$, $\pi = (\langle x_0,x_1\rangle \dots \langle x_{m-1},x_m\rangle)$, $\forall \langle x_m,y\rangle \in E(x_m)$: $\mathbf{b}(\mathbf{c}_b(\tilde{\Lambda}_{x_0}(\pi)), \tilde{\lambda}_{x_m}(\langle x_m,y\rangle)) = \mathbf{d}(\tilde{\lambda}_{x_m}(\langle x_m,y\rangle), \mathbf{c}(\tilde{\lambda}_{x_{m-1}}(\langle x_{m-1},x_m\rangle) \dots \tilde{\lambda}_{x_0}(\langle x_0,x_1\rangle)))$. ◻

The above lemmas have a very important consequence, as follows.

THEOREM 19. (G,λ) *has (weak) backward sense of direction iff* $(G,\tilde{\lambda})$ *has (weak) sense of driection.*

This result implies that the structure of the "forward" consistency landscape and that of the backward consistent are mirror images. This fact will be instrumental in simplifying the proof of some of the results to be established in the remainder of this section.

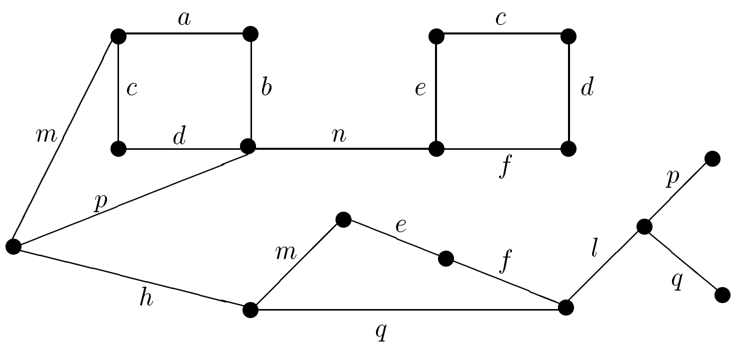

FIG. 12. $\mathbf{G}_w$.

5.2. Core. We now focus on the structure of the "core" of the consistency landscape; i.e., the set $\mathcal{W} \cap \mathcal{W}^-$.

The first result we establish is that backward sense of direction is a stronger form of consistency than weak backward sense of direction.

THEOREM 20 ($\mathcal{D}^- \subset \mathcal{W}^-$). *There are systems with backward consistency where no coding function is backward decodable.*

Proof. The proof follows from Lemmas 2 and 9. □

The next question we consider is whether the simultaneous presence of both consistencies suffices for the existence of either form of decoding. The answer is negative, as there exist systems with both weak sense of direction and weak backward sense of direction where no (backward) coding function is (backward) decodable. Let $\mathbf{G}_w$ be the labeled graph shown in Figure 12.

LEMMA 10 (see [11]). $\mathbf{G}_w \in \mathcal{W} - \mathcal{D}$.

THEOREM 21 ($(\mathcal{W} \cap \mathcal{W}^-) \not\subset (\mathcal{D} \cup \mathcal{D}^-)$). *Existence of both types of consistency is not sufficient for decodability of either type.*

Proof. By Lemma 10, graph G_w has weak sense of direction, but any coding function is not decodable. Such a labeled graph has edge-symmetry since the labeling is a coloring. Thus, by Theorem 10 we have that it also has backward weak sense of direction. However, no backward consistent coding function for G_w can be backward decodable; otherwise G_w would have backward sense of direction and, thus, by Theorem 11 would also have sense of direction contradicting Lemma 10. □

THEOREM 22 ($(\mathcal{D} \cap \mathcal{W}^-) \not\subset \mathcal{D}^-$, $\mathcal{D}^- \not\subset (\mathcal{W}^- \cap \mathcal{D})$). *Simultaneous presence of sense of direction and backward weak sense of direction is neither necessary nor sufficient for the existence of a backward decoding function.*

Proof. Not sufficient. It is easy to verify that the labeled graph of Figure 13 has sense of direction and weak backward sense of direction. However, no backward coding function is backward decodable. In fact, let $\mathbf{c}$ be a backward coding function; by definition, we have that $\mathbf{c}(dc) = \mathbf{c}(fe)$ and $\mathbf{c}(ba) = \mathbf{c}(dc)$; it follows that $\mathbf{c}(ba) = \mathbf{c}(fe)$. Let $\mathbf{b}$ be a backward decoding function; then we have that $\mathbf{b}(\mathbf{c}(ba), m) = \mathbf{c}(p) = \mathbf{b}(\mathbf{c}(fe), m)$. But $\mathbf{b}(\mathbf{c}(fe), m) = \mathbf{c}(q)$. It follows that $\mathbf{c}(q) = \mathbf{c}(p)$, which contradicts both forward and backward coding.

Not necessary. It follows from the fact that local orientation is not necessary for having a backward decoding function (see Theorem 1). □

Thus, by Theorem 19, the following holds.

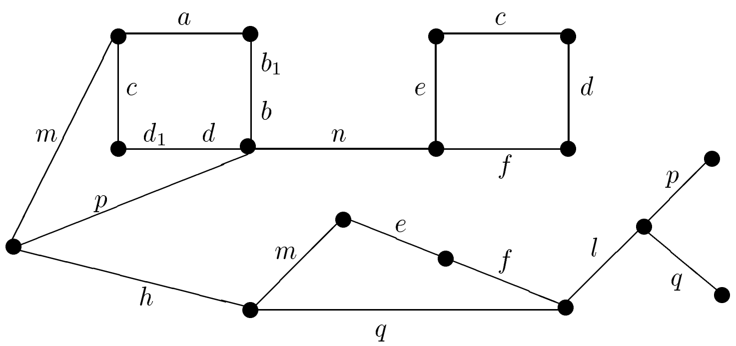

FIG. 13. $(\mathcal{D} \cap \mathcal{W}^-) - \mathcal{D}^- \neq \emptyset$.

THEOREM 23 ($(\mathcal{D}^- \cap \mathcal{W}) \not\subset \mathcal{D}$). *Simultaneous presence of backward sense of direction and weak sense of direction is not sufficient for the existence of a decoding function.*

5.3. Outer structure. In this section we focus on the outer structure of the consistency landscape: $(\mathcal{L} \cup \mathcal{L}^-) - (\mathcal{W} \cap \mathcal{W}^-)$.

We first introduce two useful properties.

Given two labeled graphs (G_1, λ_1) and (G_2, λ_2), the *melding* of (G_1, λ_1) and (G_2, λ_2) by $x_1 \in V_1, x_2 \in V_2$, denoted by $\mathbf{G_1}[x_1, x_2]\mathbf{G_2}$, is the union of the two graphs restricted by imposing $x_1 = x_2$.

THEOREM 24. *Let $\mathbf{G_1}$ and $\mathbf{G_2}$ be vertex- and label-disjoint labeled graphs with weak sense of direction. Then $\forall x_1 \in V_1, x_2 \in V_2$, $\mathbf{G} = \mathbf{G_1}[x_1, x_2]\mathbf{G_2}$ also has weak sense of direction. Furthermore, if $\mathbf{G_1}$ and $\mathbf{G_2}$ have sense of direction, (G, λ) also has sense of direction.*

Proof. By hypothesis the labeled graphs are label-disjoint, that is, $\Sigma_1 \cap \Sigma_2 = \emptyset$. By hypothesis there exists two coding function $\mathbf{c}_1$ and $\mathbf{c}_2$ such that $\mathbf{c}_1$ is a consistent coding function for G_1 and $\mathbf{c}_2$ for G_2. Without loss of generality we can also assume that $\mathcal{N}(\mathbf{c}_1) \cap \mathcal{N}(\mathbf{c}_2) = \emptyset$. (If this is not the case, take $\mathbf{c}_1' = (0, \mathbf{c}_1)$ and $\mathbf{c}_2' = (1, \mathbf{c}_2)$.) Let $i : (\Sigma_1 \cup \Sigma_2)^+ \to \{1, 2\}$ be a function such that $i(\alpha) = 1$ iff the last symbol of α belongs to Σ_1; otherwise, $i(\alpha) = 2$. Let $f : (\Sigma_1 \cup \Sigma_2)^+ \to \Sigma_1^+ \cup \Sigma_2^+$ be a function such that $f(\alpha) = \alpha \cap \Sigma_{i(\alpha)}$ (i.e., $f(\alpha)$ is the string of all the symbols of α belonging to the set $\Sigma_{i(\alpha)}$).

Let $p(x, \alpha) = x$ if $x \in G_{i(\alpha)}$, and $x_1 = x_2$ otherwise. Then it is easy to see that

$$(*) \qquad \forall \alpha \in \Lambda_G[x, y] \Rightarrow f(\alpha) \in \Lambda_{G_{i(\alpha)}}[p(x, \alpha), y].$$

Now let $\mathbf{c}$ be so defined: $\mathbf{c}(\alpha) = \mathbf{c}_{i(\alpha)}(f(\alpha))$. Now we will prove that $\mathbf{c}$ is consistent. By contradiction suppose that $\mathbf{c}$ is not consistent; then there exist $x, y, z \in V_G$ and $\alpha \in \Lambda_G[x, y], \beta \in \Lambda_G[x, z]$ such that either $\mathbf{c}(\alpha) = \mathbf{c}(\beta)$ but $y \neq z$, or $\mathbf{c}(\alpha) \neq \mathbf{c}(\beta)$ but $y = z$.

In the first case, since the codomains of $\mathbf{c}_1$ and $\mathbf{c}_2$ are disjoint, it follows that $\mathbf{c}_i(f(\alpha)) = \mathbf{c}_i(f(\beta))$ for some i such that $f(\alpha), f(\beta) \in \Sigma_i^+$. By (*) it follows that $f(\alpha) \in \Lambda_{G_i}[p(x, \alpha), y]$ and $f(\beta) \in \Lambda_{G_i}[p(x, \beta), z]$. Because of $i(\alpha) = i(\beta)$ it follows that $p(x, \alpha) = p(x, \beta)$. Thus, $f(\alpha) \in \Lambda_{G_i}[x', y]$ and $f(\beta) \in \Lambda_{G_i}[x', z]$. By consistency of c_i it follows that $y = z$, yielding a contradiction.

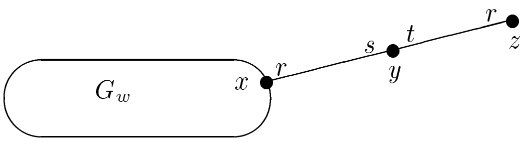

FIG. 14. $(\mathcal{W} - \mathcal{D}) - \mathcal{L}^- \neq \emptyset$.

In the second case, if $y = z$ and $y \in G_i$ for some i, by (*) and analogous deductions, we have that $f(\alpha), f(\beta) \in \Lambda_{G_i}[x', y]$ and, by consistency of $\mathbf{c}_i$, $\mathbf{c}_i(f(\alpha)) = \mathbf{c}_i(f(\beta))$; that is, $\mathbf{c}(\alpha) = \mathbf{c}(\beta)$, yielding a contradiction.

Thus, we have proved that $\mathbf{c}$ is consistent; that is, G has weak sense of direction.

Now we will prove that if G_1 and G_2 have sense of direction, then G also has sense of direction. Let $\mathbf{d}_1$ and $\mathbf{d}_2$ be the decoding functions, and let $\mathbf{d}$ be so defined:

$$\mathbf{d}(a, \mathbf{c}(w)) = \begin{cases} \mathbf{d}_1(a, \mathbf{c}(w)) & \text{if } a \in \Sigma_1 \wedge \mathbf{c}(w) \in \mathcal{N}(\mathbf{c}_1) \quad (1) \\ \mathbf{d}_2(a, \mathbf{c}(w)) & \text{if } a \in \Sigma_2 \wedge \mathbf{c}(w) \in \mathcal{N}(\mathbf{c}_2) \quad (2) \\ \mathbf{c}(w) & \text{if } a \in \Sigma_1 \wedge \mathbf{c}(w) \in \mathcal{N}(\mathbf{c}_2) \quad (3) \\ \mathbf{c}(w) & \text{if } a \in \Sigma_2 \wedge \mathbf{c}(w) \in \mathcal{N}(\mathbf{c}_1) \quad (4) \end{cases}.$$

We will show that $\mathbf{d}(a, \mathbf{c}(w)) = \mathbf{c}(aw)\ \forall\ aw \in \Lambda_G[x, z]$. We have to consider the four cases in the definition of $\mathbf{d}$.

(1) In this case $\mathbf{d}(a, \mathbf{c}(w)) = \mathbf{d}_1(a, \mathbf{c}(w))$, $a \in \Sigma_1$, and $\mathbf{c}(w) \in \mathcal{N}(\mathbf{c}_1)$. Thus, a is a label of an edge of G_1 and w is a label of a path ending in G_1. This means that $\mathbf{c}(w) = \mathbf{c}_1(f(w))$. Then, $\mathbf{d}_1(a, \mathbf{c}(w)) = \mathbf{d}_1(a, \mathbf{c}_1(f(w)))$. By property (*), it follows that $f(w)$ is the label of a path in G_1; since $x \in G_1$, $a \cdot f(w) = f(aw) \in \Lambda_{G_1}[x, z]$. By consistency of $\mathbf{d}_1$, we have $\mathbf{d}_1(a, \mathbf{c}_1(f(w))) = \mathbf{c}_1(a \cdot f(w)) = \mathbf{c}_1(f(aw)) = \mathbf{c}(aw)$.

(2) Symmetric to case (1).

(3) In this case, $\mathbf{d}(a, \mathbf{c}(w)) = \mathbf{c}(w)$, $a \in \Sigma_1$, and $\mathbf{c}(w) \in \mathcal{N}(\mathbf{c}_2)$. Thus, a is a label of an edge in G_1, while w is the labeling of a path ending in G_2. This implies that $f(aw) = f(w)$. Thus, $\mathbf{d}(a, \mathbf{c}(w)) = \mathbf{c}(w) = \mathbf{c}_2(f(w)) = \mathbf{c}_2(f(aw)) = \mathbf{c}(aw)$.

(4) Symmetric to case (3). ∎

THEOREM 25. $(\mathcal{W} - \mathcal{D}) \not\subset \mathcal{L}^-$.

Proof. Consider the labeled graph of Figure 14. Notice that such a graph is the melding in x of the graph G_w of Figure 12 and a line with two edges $\langle x, y\rangle$ and $\langle y, z\rangle$.

Since the labeled line has trivially weak sense of direction and since, by Lemma 10, the graph $\mathbf{G}_w$ of Figure 12 has weak sense of direction, it follows from Theorem 24 that the graph of Figure 14 has weak sense of direction as well.

On the other hand, by Lemma 10, the graph of Figure 12 does not have sense of direction, and, as a consequence, the graph of Figure 14 also does not have sense of direction. Moreover, this labeled graph does not have backward local orientation; in fact $\lambda_x(\langle x, y\rangle) = \lambda_z(\langle z, y\rangle)$. ∎

As a consequence, we have the following.

THEOREM 26. $(\mathcal{W}^- - \mathcal{D}^-) - \mathcal{L} \neq \emptyset$.

Proof. The proof follows from Theorems 19 and 25. ∎

THEOREM 27. $((\mathcal{W} - \mathcal{D}) \cap \mathcal{L}^-) - \mathcal{W}^- \neq \emptyset$.

Proof. Consider the labeled graph (G, λ) of Figure 15. Notice that it is the melding in x of Figure 12 and a labeled graph (G', λ'). It is easy to verify that (G', λ') has weak sense of direction, which implies that (G, λ) has weak sense of direction. However,

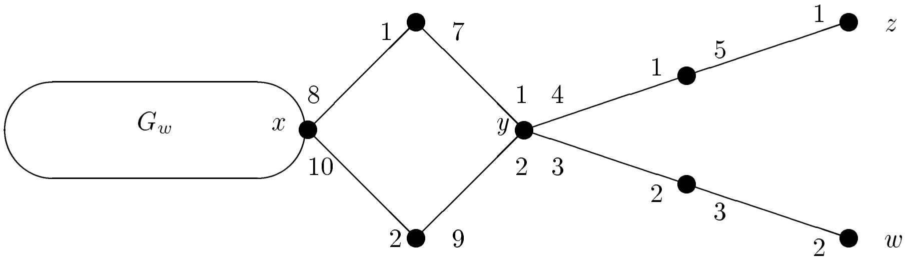

FIG. 15. $((\mathcal{W}-\mathcal{D})\cap\mathcal{L}^-)-\mathcal{W}^-\neq\emptyset$.

since by Lemma 10 the graph G_w of Figure 12 does not have sense of direction, it follows that (G,λ) also does not have sense of direction. Clearly, there is backward local orientation. By contradiction, let us assume that there is weak backward sense of direction in (G,λ). By definition of backward consistency at node x we would have that $\mathbf{c}(1\cdot 1)=\mathbf{c}(2\cdot 2)$ (since these two sequences of labels correspond to two paths from y to x); however, $\mathbf{c}(1\cdot 1)\neq\mathbf{c}(2\cdot 2)$ (since they also correspond to two paths starting from different nodes (w and z) and terminating in the same node y). ☐

As a consequence, we have the following.

THEOREM 28. *$((\mathcal{W}^- - \mathcal{D}^-)\cap\mathcal{L})-\mathcal{W}\neq\emptyset$.*

Proof. The proof follows from Theorems 19 and 27. ☐

6. Backward consistency and Σ. In this section we ask some questions about the relationship between the set of labels used in (G,λ) and the existence of backward consistency. In particular, we focus on the number $\kappa(G,\lambda)=|\Sigma|$ of labels used in (G,λ). A general bound on κ is the following, where $\deg(G)$ and $e(G)$ denote the maximum degree and the number of edges in G, respectively.

LEMMA 11. *If (G,λ) has backward consistency, then $d(G)\leq\kappa(G,\lambda)\leq 2e(G)$.*

6.1. Minimum backward sense of direction. If $\kappa(G,\lambda)=d(G)$, the labeling is said to be *minimum* and so is called any consistency property existing in (G,λ). So, consistency in minimally labeled systems is called *minimum (weak) sense of direction*; similarly we can define the backward analogues. Let $m\mathcal{C}$ denote the subclass of $\mathcal{C}$ of labeled graphs (G,λ) for which λ is a minimum labeling (e.g., $m\mathcal{W}$ is the class of labeled graphs with minimum weak sense of direction).

We now study the relationship between minimality and biconsistency in arbitrary topologies.

THEOREM 29. *Edge-symmetry in a minimum weak sense of direction is not sufficient for backward consistency.*

Proof. Consider the labeled graph of Figure 16.

For each $\alpha\in\{r,l,a\}^*$, let $\mu(\alpha)\in\{r,a\}^*$, the string obtained from α applying as many times as possible one of the following simplification rules: $rl\to\epsilon$, $lr\to\epsilon$, $aa\to\epsilon$, $rrr\to\epsilon$, $lll\to\epsilon$, $l\to rr$. The first three rules eliminate all the cycles of length 2, and the following two rules eliminate all the cycles of length 3. The last rule rewrites the l with two r's. Let $A=\{\alpha:\exists v\wedge\exists\pi\in P[v]:\Lambda_v(\pi)=\alpha\}$, and let $\mu(A)=\{\mu(\alpha):\alpha\in A\}$. It can be proven that $\mu(A)=r^{2*}(\cdot a\cdot r^{2*})^?$, where r^{2*} means at most 2 repetitions of the symbol r (there can be 0 repetitions). $(x)^?$ means 0 or 1 repetition of x. Let a coding function c be defined as follows:

$$c(\alpha)=\begin{cases}(\mu(\alpha),0) & \text{if } \mu(\alpha)\in r^{2*},\\ (\beta,1)) & \text{if } \mu(\alpha)=r^{2*}a\beta.\end{cases}$$

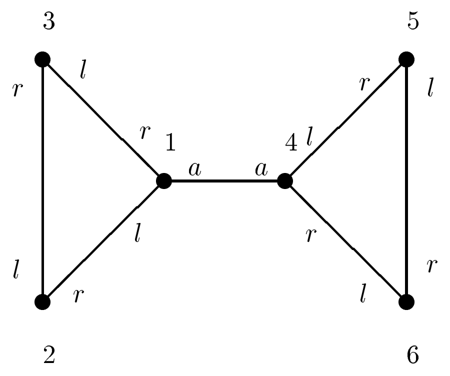

FIG. 16. *A minimum labeled graph with a consistent coding c that is not backward consistent.*

For each node v let $\overline{v}$ be the node on the same ring with degree 3 (e.g., $\overline{2} = \overline{3} = \overline{1} = 1$ and $\overline{4} = \overline{5} = \overline{6} = 4$). We can prove the following property.

CLAIM 2.

1. $\forall x, y, \pi \in P[x, y], (\mathbf{c}(\Lambda_x(\pi)) = \mathbf{c}(\gamma, 0) \Leftrightarrow \overline{x} = \overline{y} \wedge \exists \pi' \in P[x, y] \wedge \Lambda_x(\pi') = \gamma)$.
2. $\forall x, y, \pi \in P[x, y], (\mathbf{c}(\Lambda_x(\pi)) = \mathbf{c}(\gamma, 1) \Leftrightarrow \overline{x} \neq \overline{y} \wedge \exists \pi' \in P[\overline{y}, y] \wedge \Lambda_{\overline{y}}(\pi') = \gamma)$.

It is easy to see that $\mathbf{c}$ is a consistent coding function. But $\mathbf{c}$ is not backward consistent. We can see that ra is the label of the path from 6 to 1 passing from node 4, and a is the label of the path from 4 to 1. Thus $4 \neq 6$, but $\mathbf{c}(ra) = (\epsilon, 1) = \mathbf{c}(a)$. □

COROLLARY 1. *Edge-symmetry in a minimum (backward) weak sense of direction is not sufficient for biconsistency.*

Note that this also gives, as a corollary, a different proof of Theorem 13.

Now we will study the relationship between sense of direction and minimality in regular graphs. Let $\mathcal{R}$ denote the set of labeled regular graphs, that is, $\deg(v) = k$ for all nodes v.

First observe that the requirement of minimality does not make edge-symmetry necessary for either form of consistency.

LEMMA 12 ($\mathcal{R} \cap m\mathcal{W}^{(-)} \not\subset \mathcal{ES}$). *Edge-symmetry is not necessary for having minimum weak sense of direction or minimum weak backward sense of direction in regular graphs.*

Proof. The nonnecessity for minimum weak backward sense of direction can be seen by considering Figure 1, where the graph has minimum weak backward sense of direction but does not have edge-symmetry; the mirror image result follows by Theorem 19. □

However, for the simultaneous existence of both types of consistency, unlike the general case (see Theorem 12), edge-symmetry is necessary if minimality is required.

THEOREM 30 ($m\mathcal{W}^{\pm} \cap \mathcal{R} = m\mathcal{W} \cap \mathcal{ES} \cap \mathcal{R} = m\mathcal{W}^{-} \cap \mathcal{ES} \cap \mathcal{R}$). *In a regular labeled graph (G, λ) with edge-symmetry, any minimum weak sense of direction (minimum weak backward sense of direction) is also biconsistent.*

Proof. Let $\mathbf{c}$ be a consistent coding function for (G, λ). By contradiction, let us assume that $\mathbf{c}$ does not have backward consistency, which means that there exist two paths $\pi_1 \in P[x, z], \pi_2 \in P[y, z]$ such that either (i) $\mathbf{c}(\Lambda_x(\pi_1)) = \mathbf{c}(\Lambda_y(\pi_2))$ and $x \neq y$ or (ii) $\mathbf{c}(\Lambda_x(\pi_1)) \neq \mathbf{c}(\Lambda_y(\pi_2))$ and $x = y$.

Let us consider case (i). Since the graph is regular and the labeling minimum, there must exist a path $\pi_3 \in P[x]$ such that $\Lambda_x(\pi_3) = \Lambda_y(\pi_2)$. Thus, $\mathbf{c}(\Lambda_x(\pi_1)) =$

$\mathbf{c}(\Lambda_y(\pi_3))$, which implies that $\pi_3 \in P[x,z]$. This is a contradiction, because it violates local orientation in z.

Consider now case (ii). This case violates the definition of forward consistency in x.

The reverse is analogous. □

Thus, with edge-symmetry, the class of regular graphs with minimum weak backward sense of direction coincides with the class of edge-symmetric regular graphs with minimum weak sense of direction. Using the characterization of the latter class by [10, 23], we obtain the following.

THEOREM 31. *A regular labeled graph (G,λ) with edge-symmetry has minimum weak backward sense of direction iff it is a Cayley graph with Cayley labeling.*

In other words, if there is edge-symmetry, Cayley graphs with Cayley labelings are the only regular graphs with minimum backward sense of direction and/or minimum sense of direction.

COROLLARY 2 ($m\mathcal{W}^{\pm} \cap \mathcal{ES} \cap \mathcal{R} = m\mathcal{D}^{\pm} \cap \mathcal{ES} \cap \mathcal{R}$). *In a regular labeled graph with edge-symmetry, any minimum and biconsistent weak sense of direction is a sense of direction.*

The situation is quite different in the absence of edge-symmetry; in this case, at most one type of minimum consistency can exist.

LEMMA 13 ($m\mathcal{W} \cap m\mathcal{W}^{-} \subset \mathcal{ES}$). *Without edge-symmetry and restricted to regular graphs, $m\mathcal{W} \cap m\mathcal{W}^{-} = \emptyset$.*

Proof. The proof follows from Theorem 30. □

6.2. Number of labels and backward consistency. In this section we ask under what conditions a (large) value $\kappa(G,\lambda)$ is sufficient for backward consistency. The first obvious result is that $\kappa(G,\lambda) = 2e(G)$; then the system has both sense of direction and backward sense of direction without further conditions.

LEMMA 14. *If $\kappa(G,\lambda) = 2e(G)$, then (G,λ) has both sense of direction and backward sense of direction.*

Basically, this numerical requirement is equivalent to having all edges labeled with unique labels.

If we are interested only in backward sense of direction, we can actually relax this condition. In fact, it is sufficient that the labels used by each node are different from the ones used by the other nodes for the system to have backward sense of direction (but not necessarily sense of direction); see, for example, Figure 17. Let $\Sigma_x = \{\lambda_x(x,y) : y \in E(x)\}$.

THEOREM 32. *If, in (G,λ), $\Sigma_x \cap \Sigma_y = \emptyset \; \forall \; x \neq y$, then (G,λ) has backward sense of direction.*

Proof. For each $a \in \Sigma$, only one node x uses a: let $\sigma(a) = \Sigma_x$.

Consider the following backward coding function: $\forall \pi \in P[x_0]$, $\pi = \langle x_0, x_1\rangle \ldots \langle x_{m-1}, x_m\rangle$, $\mathbf{c}(\Lambda_x(\pi)) = \sigma(\lambda_{x_0}(\langle x_0, x_1\rangle))$.

Let $\pi_1 \in P[x,z], \pi_2 \in P[y,z]$; $\pi_1 = \langle x, x_1\rangle \ldots \langle x_h, z\rangle$, $\pi_2 = \langle y, y_1\rangle \ldots \langle y_k, z\rangle$. We have that $\mathbf{c}(\Lambda_x(\pi_1)) = \sigma(\lambda_x(\langle x, x_1\rangle))$ and $\mathbf{c}(\Lambda_y(\pi_2)) = \sigma(\lambda_y(\langle y, y_1\rangle))$.

Consider the case $x = y$. By definition of σ, we have that $\sigma(\lambda_x(\langle x, x_1\rangle)) = \sigma(\lambda_y(\langle y, y_1\rangle))$; thus, $\mathbf{c}(\Lambda_x(\pi_1)) = \mathbf{c}(\Lambda_y(\pi_2))$.

Consider the case $x \neq y$. Since, by hypothesis, all the labels used by x are different from all the labels used by y, we have $\sigma(\lambda_x(\langle x, x_1\rangle)) \neq \sigma(\lambda_y(\langle y, y_1\rangle))$ and $\mathbf{c}(\Lambda_x(\pi_1)) \neq \mathbf{c}(\Lambda_y(\pi_2))$.

It is easy to see that the corresponding backward consistent decoding function is

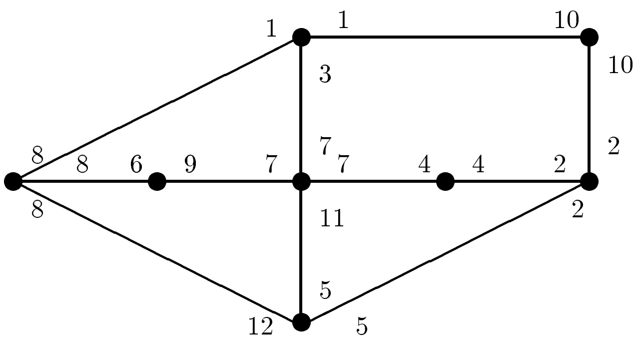

FIG. 17. $\Sigma_x \cap \Sigma_y = \emptyset$.

the following:

$$\forall \pi \in P[x_0], \pi = \langle x_0, x_1 \rangle \ldots \langle x_{m-1}, x_m \rangle, \forall \langle x_m, y \rangle \in E(x_m),$$

$$\mathbf{b}(\mathbf{c}(\Lambda_{x_0}(\pi), \lambda_{x_m}(\langle x_m, y \rangle)) = \mathbf{c}(\Lambda_{x_0}(\pi)). \qquad \square$$

Analogously, when the labels used for the edges terminating in x are different from the ones used for edges terminating in any $y \neq x$, then the system has sense of direction. Let $\Sigma_x^- = \{\Sigma_y(y, x) : y \in E(x)\}$.

THEOREM 33. *If, in* (G, λ), $\Sigma_x^- \cap \Sigma_y^- = \emptyset \ \forall \ x \neq y$, *then* (G, λ) *has sense of direction.*

Notice that among the labelings satisfying the assumption of Theorem 32 are those yielding the systems with complete blindness discussed in Theorem 2; among the labelings satisfying the assumption of Theorem 33 are the so-called neighboring ones. In both cases the number of labels used is exactly $n(G)$.

7. Computational equivalence. ¿From a computational viewpoint, *anonymous* systems are the least powerful distributed systems—and hence the ideal setting in which to study the capabilities of specific properties (e.g., [1, 2, 7, 16, 22, 35, 36, 38, 39, 43, 44]). The computational power of sense of direction in anonymous systems has been investigated for specific labeled graphs (e.g., meshes and tori [4, 34], hypercubes [28, 29], and Cayley graphs [27]). The general characterization has been given in [22]; as an indication of its capabilities, many unsolvable problems in anonymous networks (e.g., computing the XOR in a regular network without knowledge of the network size) can be solved if the system has sense of direction (and without breaking anonymity). To exist, and thus to be exploited, sense of direction requires the existence of constraints much stronger than local orientation.

What happens, from a computational point of view, if the system does not even guarantee local orientation? What can be computed if nodes cannot even distinguish among their links? There are several studies on computability in the absence of local orientation investigating these questions for specific topologies [14], specific problems [45], or classes of functions [9, 37]. As expected, these results indicate that the absence of local orientation dramatically reduces the computational power in almost all graphs (a noticeable exception is the ring [14]).

In this section we study the computational capability of the newly introduced backward consistency, which can exist even in the presence of complete and total blindness. We show the unexpected result that *backward consistency has the same computational capabilities as sense of direction.* In other words, not only does it overcome the handicap of not having local orientation, but it also actually empowers even a totally blind system with the additional capabilities of sense of direction.

7.1. Computability. A basic concept when computing on anonymous networks is the one of *view*, introduced in [44]. The *view* $T_{(G,\lambda)}(v)$ of a node v in a labeled graph (G, λ) is an infinite, labeled, rooted tree "downward locally isomorphic" to G, i.e., such that there exists a map from the vertices of the tree to the vertices of G which maps the root of the tree to v, the children of the root to the neighbors of v, and, recursively, the children of a node to the neighbors of that node. The labeling of the arcs are also downward preserved. When no ambiguity arises, we shall denote a view $T_{(G,\lambda)}(v)$ simply by $T(v)$.

Let $\mathcal{G}$ denote the set of labeled graphs. Given $\mathcal{C} \subseteq \mathcal{G}$ and a graph G, let $\mathcal{C}/G$ denote the restriction of $\mathcal{C}$ to G; e.g., $\mathcal{L}^-/G$ is the set of labeled graphs obtained by labeling G with backward local orientations.

A problem $\mathcal{P}$ is *solvable* in $\mathcal{C}/G$ if it is solvable in all labeled graphs $(G, \lambda) \in \mathcal{C}/G$. Given a topological knowledge K, a problem $\mathcal{P}$ is K-solvable in $\mathcal{C}/G$ if it is solvable in all $(G, \lambda) \in \mathcal{C}/G$, where all the nodes are empowered with a priori knowledge K.

We first report two results on the computational capabilities of sense of direction. The first result is that, from a computational point of view, there is no difference between weak sense of direction and sense of direction.

THEOREM 34 (see [22]). *$\forall G \forall \mathcal{P}$, $\mathcal{P}$ is solvable in $\mathcal{W}/G$ iff $\mathcal{P}$ is solvable in $\mathcal{D}/G$.*

The second result shows that, with sense of direction, no other knowledge is necessary.

THEOREM 35 (see [22]). *$\forall G \forall \mathcal{P} \forall K$, if $\mathcal{P}$ is K-solvable in $\mathcal{L}/G$, then $\mathcal{P}$ is solvable in $\mathcal{W}/G$.*

We will now focus on backward consistency and establish the equivalence result.

THEOREM 36. *$\forall G \forall \mathcal{P}$, $\mathcal{P}$ is solvable in $\mathcal{D}/G$ iff $\mathcal{P}$ is solvable in $\mathcal{D}^-/G$.*

Proof. To prove this theorem we will need several steps.

Given two labeled graphs $(G = (V, E), \lambda)$ and $(G' = (V', E'), \lambda')$, a *labeled graph isomorphism* is a bijection $\chi : V \to V'$ which preserves edges and edge labels; that is, $\langle u, v\rangle \in E \Leftrightarrow \langle \chi(u), \chi(v)\rangle \in E'$ and $\lambda(\langle u, v\rangle) = \lambda'(\langle \chi(u), \chi(v)\rangle)$.

The *complete topological awareness* of a node $x \in V$ in a labeled graph $(G = (V, E), \lambda)$ is knowledge of the isomorphic image $\psi(G, \lambda)$ of (G, λ) and of $\phi(x)$ for some labeled graph isomorphism ψ. The *complete topological knowledge* on a labeled graph $(G = (V, E), \lambda)$ (denoted by TK) is when every node $x \in V$ has complete topological awareness. Note that the isomorphism known by each node may be different.

The importance of TK becomes apparent with the following fact.

LEMMA 15 (see [22]). *$\forall G \forall \mathcal{P}$, $\mathcal{P}$ is TK-solvable in $\mathcal{L}/G$ iff $\mathcal{P}$ is solvable in $\mathcal{D}/G$.*

In other words, TK represents the maximum information obtainable with sense of direction. Hence, to prove the main theorem, we need only show that every node can construct TK with weak backward sense of direction. To do this, first observe the following.

LEMMA 16 (see [22]). *Let $\mathbf{c}$ be a consistent coding function of (G, λ). Then $\forall u \in V$, $T_{(G,\lambda)}/\mathbf{c} = \psi(G, \lambda)$ for some labeled graph isomorphism ψ.*

That is, consistency would allow each node to construct an isomorphic image of (G, λ) from each view. Since in its view a node knows its location (it is the root), it knows its location in the image. Therefore, to complete the proof, we have to show how to constructively empower each node with local orientation and a consistent coding.

If there is backward consistency in (G, λ), a local orientation with forward consistency can be easily constructed by employing the result of Lemma 9: if (G, λ) has backward consistency, then $(G, \tilde{\lambda})$ has local orientation and consistency. Since $\tilde{\lambda}$ is distributedly constructible (i.e., each node x can construct $\tilde{\lambda}_x$) by a simple round of communication), $T_{(G,\tilde{\lambda})}(u)$ is also constructible at every node.

Thus, we must only show how to construct at each node a consistent coding function $\mathbf{c}$ for $(G, \tilde{\lambda})$, given $\mathbf{b}$, λ_x, and $\tilde{\lambda}_x$. This is done by observing that, by Theorem 18, if (G, λ) has backward consistent coding $\mathbf{b}$, then we can construct a consistent coding $\mathbf{c}$ of (G, λ^2), and by Lemma 9, $\mathbf{c}$ is a consistent coding for $(G, \tilde{\lambda})$.

Summarizing, if (G, λ) has backward consistency $\mathbf{b}$, each node v can construct the view $T_{(G,\tilde{\lambda})}(v)$ and a consistent coding of $(G, \tilde{\lambda})$ which will empower it to construct both an isomorphic image of both (G, λ) and $\psi(v)$. This knowledge is equivalent to complete topological knowledge; since the class of problems solvable with TK is exactly the same solvable with sense of direction alone, it follows that backward sense of direction is computationally equivalent to sense of direction. □

7.2. Complexity. The result of Theorem 36 opens the problem of how to effectively use the computational power of backward consistency. The approach suggested by the proof of Theorem 36 is to use backward consistency to "simulate" sense of direction. While theoretically valid, this approach sidesteps the problem since it does not exploit backward consistency directly. Furthermore, the technique employed in the proof is not algorithmically reasonable, since it requires the construction of the views $T_{(G,\tilde{\lambda})}(v)$, a task with a formidable communication complexity.

While not solving the general problem, we will provide some complexity relief by pointing out how the simulation approach can be carried out simply and efficiently, without construction of the views.

Let A be an algorithm which solves problem $\mathcal{P}$ in any system with sense of direction. The simulation $S(A)$ of A in a system (G, λ) in which there is backward sense of direction is performed in two stages:

(1) *Preprocessing.* Each processor x computes the set $\Sigma_x(a) = \{b : \lambda_x(x, y) = a$ and $\lambda_y(y, x) = b\}$ for each of its labels $a \in \lambda_x(E(x))$. Notice that $\bigcup \Sigma(a) = \tilde{\lambda}_x(E(x))$.

(2) *Simulation.* Whenever, in algorithm A, a node x sends a message m on an edge labeled l, in the simulation $S(A)$ it will send a message (m, l) to the set of edges labeled by the (unique) label $p \in \lambda_x(E(x))$ such that $l \in \Sigma_x(p)$; whenever, in algorithm A, a node performs an action O upon reception of a message m from a link labeled l, in the simulation of A it will perform the same action O upon reception of message (m, l) from any of the edges labeled p, where p is the (unique) label such that $l \in \Sigma_x(p)$.

It is easy to see that algorithm $S(A)$ behaves on (G, λ) in exactly the same way as the algorithm A would behave on $(G, \tilde{\lambda})$ (the only difference being that $S(A)$ sends a pair $(m, label)$ every time A sends a message m). In fact, we have the following result.

THEOREM 37. *Algorithm $S(A)$ solves $\mathcal{P}$ in any system with backward sense of direction iff A solves $\mathcal{P}$ in any system with sense of direction.*

Proof. Let A solve P in any system with sense of direction. Since (G, λ) has backward sense of direction, from Theorem 19, we have that $(G, \tilde{\lambda})$ is a system with sense of direction; and thus A solves P on $(G, \tilde{\lambda})$. But, by construction, algorithm S behaves on (G, λ) in exactly the same way as the algorithm A would behave on $(G, \tilde{\lambda})$, and the proof follows.

The reverse is analogous. □

Moreover, the number of message *transmissions* of $S(A)$ (denoted by $\mathcal{M}_T$) is the same as in the original algorithm A. The number of message *receptions* (denoted by $\mathcal{M}_R$) obviously depends on the cardinality of the sets $\Sigma_x(p)$. Given (G, λ), let $h(G, \lambda) = Max_{x \in V, a \in \Sigma}\{\Sigma_x(a)\}$; clearly, $h(G, \lambda) \leq d(G)$.

THEOREM 38.

(1) $\mathcal{M}_T(S(A), G, \lambda) = \mathcal{M}_T(A, G, \tilde{\lambda})$.

(2) $\mathcal{M}_R(S(A), G, \lambda) \leq h(G, \lambda)\mathcal{M}_R(A, G, \tilde{\lambda})$.

In other words, it is possible to use simulation with some level of efficiency. However, the real task is to develop protocols and techniques which exploit backward consistency directly (not just to simulate forward consistency).

8. Concluding remarks and open problems. Sense of direction, for its existence, requires the presence of many "low-level" subconditions, in particular local orientation. However, local orientation cannot be assumed if we are to model systems with more advanced communication and interconnection technology. In this paper we have shown the existence of another type of consistency—backward consistency—which is computationally equivalent to sense of direction but does not require local orientation; thus, it can be found (or designed) in advanced distributed systems.

The relationship between backward consistency and communication complexity is still virtually unknown. The study of how to effectively and efficiently exploit the power of backward consistency when solving problems is an open research area. Some results already exist for specific problems; see, for example, the protocols developed for multi-hop radio networks [3, 12].

Sense of direction and backward consistency are two among a large (and practically unknown) population of varying types of general consistency. An interesting aspect of the results presented here is that backward consistency and sense of direction are not comparable, but rather orthogonal. This fact raises the possibility of the existence of yet other types of consistency that are computationally equivalent to, but not comparable to, sense of direction.

There are interesting coincidences and analogies; for example, the discovery that the reverse of labeling with an important property (sense of direction) has another important property (backward consistency) finds its parallel in the context of interval routing, where it was recently discovered that the reversal of any interval-routing labeling is a broadcast labeling [13]. This coincidence reveals a new viewpoint from which to examine the problem: any operation on the labeling will transform the class of problem solvable in the resulting system (the result will obviously depend on the original type of consistency). The open questions then become, What operations on the labeling (such as reversal) preserve computability? Which will transform it? And into what will it be transformed?

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